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Operator Theory
Advances and Applications
Vol. 193

Analysis, Partial Differential Equations and Applications

The Vladimir Maz'ya Anniversary Volume

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On the Occasion of the 70th Birthday of Vladimir Maz'ya

Alberto Cialdea, Flavia Lanzara and Paolo E. Ricci

This volume includes a selection of lectures given at the International Workshop "Analysis, Partial Differential Equations and Applications", held at the Mathematical Department of Sapienza University (Rome, June 30th–July 3rd, 2008), on the occasion of the 70th birthday of Vladimir Maz'ya.

Besides Italy, twenty seven countries were represented there: Belarus, Canada, China, Colombia, Croatia, Czech Republic, Finland, France, Georgia, Germany, Greece, Israel, Mexico, New Zealand, Poland, Portugal, Rumania, Russia, Saudi Arabia, South Korea, Spain, Sweden, Taiwan, The Netherlands, Turkey, United Kingdom, and United States of America.

It is not surprising that the decision of the Italian National Institute for Advanced Mathematics "F. Severi" (INDAM) to dedicate a Workshop to Vladimir Maz'ya was crowned by such great success. The scientific and human endowments of Maz'ya are well known.

He has inspired numerous researchers in Analysis and its applications, among them many in Italy. Maz'ya gladly acknowledges that this inspiration has been mutual. The Italian school of Analysis and PDEs has played an important role in his development, starting with his undergraduate years 1955–1960 and continuing to this day. As a third year student, through S. Mikhlin's lectures, he became acquainted with Tricomi's pioneering work on multi-dimensional singular integrals [37], [38], a topic of Maz'ya's keen interest in the future ([14], [21] and others). A year later, Vladimir discovered the equivalence of various Sobolev type inequalities with isoperimetric and isocapacitary inequalities, which strongly influenced functional analysis and partial differential equations in subsequent years. In particular, he found the sharp constant in the E. Gagliardo inequality between the $L_{n/(n-1)}$ norm of a function and the L_1 norm of its gradient [10]. Later Gagliardo's results on boundary traces of Sobolev functions were developed by Maz'ya and his colleagues in various directions (see, for example, [24], [33], [34]).

Following Mikhlin's recommendation, Maz'ya read the Russian translation of Carlo Miranda's "Equazioni alle Derivate Parziali di Tipo Ellittico" [36], which had appeared in 1957 in Moscow. This comprehensive survey of the Italian contribution to the field, which at that time was undergoing a major expansion, became the first book on PDEs to be read by the young Maz'ya. Miranda's book strongly influenced the shaping of Vladimir's professional interests. An evidence to this is his first publication which appeared in [9] exactly 50 years ago.

The year 1957 saw the appearance of the seminal article by E. De Giorgi on the Hölder regularity of solutions to elliptic second-order equations with measurable bounded coefficients, which had a tremendous impact on the theory of PDEs, not least the work of Maz'ya. In the article [11] of 1961, he solved a problem posed by G. Stampacchia on an estimate of weak solutions to the equations just mentioned. One of the original traits of this short paper was a characterization of the boundary in terms of an isoperimetric function introduced by the author, which enabled him to study the sharp dependence of the regularity properties of solutions to the Neumann problem on the behaviour of the boundary. A detailed exposition of this work, containing a wealth of new ideas, was published in [18], 1969.

In [12], 1963, Maz'ya obtained his famous estimate of the continuity modulus of a solution to the Dirichlet problem near a boundary point, formulated in terms of the Wiener integral (see also [15], [16]). Later, a result of the same nature was obtained by him for nonlinear equations including the p-Laplacian [19]. It is noteworthy that the classical paper by Littman, Stampacchia and Weinberger [8] on the Wiener regularity of a boundary point was translated into Russian by Maz'ya for the Moscow collection of translations "Matematika" from a preprint, even before its publication in a journal.

Of exceptional importance were Maz'ya's counterexamples relating to the 19th and 20th Hilbert problems for higher-order elliptic equations which appeared in [17], 1968, independently of and simultaneously with analogous counterexamples of E. De Giorgi and E. Giusti–M. Miranda.

The results of L. Cesari, R. Caccioppoli and especially E. De Giorgi on generalization of the notion of the surface area on nonsmooth surfaces played an important role in the pioneering research of Maz'ya and his coauthors in the theory of harmonic potentials on nonsmooth domains as well as in the theory of spaces of functions with bounded variation [1], [13], [2], [3].

The influence of G. Cimmino's results of 1937 [4] on the Dirichlet problem with boundary data in L_p as well as G. Fichera's unified theory of elliptic-parabolic equations [5] can be traced in Maz'ya's breakthrough work on the generic degenerating oblique derivative problem [20].

One of the fundamental results in the theory of partial differential equations, the C. Miranda–Sh. Agmon maximum principle for higher-order elliptic equations, was crucially developed by Maz'ya and his collaborators in several directions: polyhedral domains [23], sharp constants [22], parabolic systems [7].

The above, by necessity a rather incomplete survey, clearly shows that the Italian school stimulated the early work of Maz'ya in spite of the iron curtain. With time the contacts became bilateral and even personal. At the moment, Maz'ya is collaborating with a number of Italian mathematicians which can be seen, for instance, in some papers included into the present volume.

It is impossible in this short article to recall all Maz'ya's important achievements. In order to give an impression of the phenomenal variety of his results and without aiming at completeness we would like only to list certain fields he contributed to:

1. Equivalence of isoperimetric and integral inequalities
2. Theory of capacities and nonlinear potentials
3. Counterexamples related to the 19th and 20th Hilbert problems
4. Boundary behaviour of solutions to elliptic equations in general domains
5. Non-elliptic singular integral and pseudodifferential operators
6. Degenerating oblique derivative problem
7. Estimates for general differential operators
8. The method of boundary integral equations
9. Linear theory of surface waves
10. The Cauchy problem for the Laplace equation
11. Theory of multipliers in spaces of differentiable functions
12. Characteristic Cauchy problem for hyperbolic equations
13. Boundary value problems in domains with piecewise smooth boundaries
14. Asymptotic theory of differential and difference equations with operator coefficients

Accademia Nazionale dei Lincei, Rome. From left to right: Ennio De Giorgi, Gaetano Fichera, Vladimir Maz'ya and Giorgio Salvini (President of the Accademia).

15. Maximum modulus principle for elliptic and parabolic systems, contractivity of semigroups
16. Iterative procedures for solving ill-posed boundary value problems
17. Asymptotic theory of singularly perturbed boundary value problems
18. "Approximate approximations" and their applications
19. Wiener test for higher-order elliptic equations
20. Spectral theory of the Schrödinger operator
21. Navier-Stokes equations
22. History of Mathematics

On the occasion of Maz'ya's 60th birthday, two international conferences were held, at the University of Rostock in 1998 and at the École Polytechnique in Paris in 1998. We mention also the Nordic-Russian Symposium which was held in Stockholm in honor of his 70th birthday in 2008.

The initiative to dedicate an INDAM Workshop to Vladimir Maz'ya came from the authors of this paper, former students of Gaetano Fichera. Maz'ya and Fichera first met in the USSR in the early seventies. A story of their friendship and mathematical interaction was recounted in Maz'ya's article [25]. Together they wrote an article in honor of S. Mikhlin on the occasion of his birthday in 1978 [6]. Because of Maz'ya's ability to give complete solutions to problems which are generally considered as unsolvable, Fichera once compared Maz'ya with Santa Rita, the 14th century Italian nun who is the Patron Saint of Impossible Causes.

We are sure that Vladimir Maz'ya has kept the energy of his younger age, and after nearly thirty published volumes and more than four hundred scientific articles, he is able to deal with his "impossible" problems. During the last decade he published five new books (see [B2], [B4], [B5], [B8], [B9] in the list of Maz'ya's books), and more than 130 papers. In particular, recently Maz'ya obtained a breakthrough necessary and sufficient condition of Wiener type for regularity of a boundary point for higher-order elliptic equations [26], he had found several deep analytic criteria in the spectral theory of second-order differential operators [27], [30], [31], and solved a long-standing Gelfand's problem concerning the discreteness criterion for the Schrödinger operator [28]. He has also found sharp two-sided estimates for the first eigenvalue of the Laplacian formulated in terms of the capacitary interior diameter [29], obtained a new class of uniform asymptotic approximations of Green's kernels for singularly perturbed domains [32] and proposed an ingenious method for the asymptotic treatment of boundary value problems in perforated domains [35]. A joint book with A. Soloviev on boundary integral equations in domains with peaks will be published soon by Birkhäuser [B1] and an extended version of Maz'ya's classical monograph on Sobolev spaces will appear in Springer.

We congratulate Vladimir Maz'ya on his birthday and wish him every joy, happiness and great fulfillment in the years to come.

Before concluding this paper we would like to thank all the public and private bodies who made this event possible with their generous support.

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- The Research Project "Funzioni speciali multidimensionali e applicazioni a problemi della Fisica Matematica classica", Coordinator Paolo E. Ricci, Sapienza University of Rome.

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Boundary Trace for BV Functions in Regions with Irregular Boundary

Yuri Burago and Nikolay N. Kosovsky

To Vladimir Maz'ya on the occasion of his 70th birthday

Abstract. The aim of this work is to generalize some results of [2] by Yu. Burago and V. Maz'ya for a wider class of regions with irregular boundaries.

Mathematics Subject Classification (2000). Primary 46-E35; Secondary 28-A75.

Keywords. Embedding theorems, perimeter, rectifiability, trace.

1. Introduction

The aim of this paper is to generalize some results of [2] (also see [5] for a more detailed exposition). Namely, we generalize results on boundary trace for functions of the class BV for a wider class of regions Ω in $\mathbb{R}^n$ with irregular boundaries. In [2], [5] boundary traces were defined and studied for regions with finite perimeters and under the additional assumption that Federer's normals exist almost everywhere on the boundary $\partial\Omega$. Here we assume that $\partial\Omega$ is an $(n-1)$-rectifiable set, which is a more general condition.

We summarize some basic properties of rectifiable sets in the next subsection.

To explain why our results generalize those in [2], let us consider an open disk in $\mathbb{R}^2$ with a sequence of intervals I_i removed. The results of [2] on boundary traces of BV functions are not applicable to such a region Ω whereas our assumptions are satisfied for this region provided that the sum of lengths of the intervals is finite. Note that even for a smooth function on Ω its limits at a point of I_i from right and left can be different. So it is reasonable to introduce traces with two different values in some points.

The analytical tools we use here are basically the same as in [2]. They go back to the pioneering work by V. Maz'ya [6] where, among other results, connection

This work was partially supported by RFBR grant 08-01-00079a.

between isoperimetric inequalities and integral inequalities (of Sobolev embedding theorems type) was established.

In this paper we give generalized versions of only the key theorems from [2], [5]. We believe that there are other results on boundary traces of BV functions that can be generalized to a more general set-up. We plan to address them elsewhere.[1]

By Corollary 2.11 the main definitions and results of this paper can be generalized from $\mathbb{R}^n$ to an arbitrary C^1-smooth manifold.

Rectifiable sets. We begin with recalling the definition and basic properties of k-rectifiable sets. For more detailed exposition and proofs see [3], Chapter 3, and more specifically 3.2.19, 3.2.25, 3.2.29.

Let H_k denote the k-dimensional Hausdorff measure. A measurable set in $\mathbb{R}^n$ is said to be countable (k, H_k)-rectifiable if, up to a set of H_k-measure zero, if it is a union of a finite or countable set of images of bounded sets under Lipschitz maps in $\mathbb{R}^k$. For brevity we call countable (k, H_k)-rectifiable sets k-rectifiable. We always assume that k-rectifiable sets are H_k-measurable and that the intersection of such a set with any compact set has finite H_k-measure.

We call a point x of a k-rectifiable set A *regular* if H_k-approximative tangent cone $T_x = \mathrm{Tan}_x^k(A)$ is a k-dimensional plane.

For a k-rectifiable set A the following properties hold:

(1) Almost all (with respect to the measure H_k) points of A are regular.
(2) The map $\phi\colon A \to G_{n,k}$, where $\phi(x) = T_x$, is measurable. Here $G_{n,k}$ is the Grassman manifold of nonoriented k-planes.
(3) The density $\Theta_p^k(A) = 1$ almost everywhere (by Hausdorff measure H_k); here

$$\Theta_p^k(A) = \lim_{r\to 0} v_k r^{-k} H_k(A \cap B_p(r)),$$

where v_k is the volume of the unit ball in $\mathbb{R}^k$.
(4) Denote by $A\triangle B$ the symmetric difference $(A \setminus B) \cup (B \setminus A)$. There exists a countable family of C^1-smooth k-submanifolds $M_i \subset \mathbb{R}^n$ such that $\cup M_i$ contains H_k-almost all points of A and for almost every point p of A there exists a submanifold M_i such that T_x coincides with the tangent plane to M at x, and k-density $\Theta_p^k(A\triangle M) = 0$.

Notations. From now on we suppose that $\partial\Omega$ is a $(n-1)$-rectifiable set. $\mathrm{Vol}\, A$ denotes the n-dimensional Hausdorff measure of A or, equivalently, its Lebesgue measure in $\mathbb{R}^n$. The dimension $k = n-1$ plays a special role for us and to be short we denote $H_{n-1} = \mu$. From here on we use terms “almost all”, “measurable”, etc, with respect to a Hausdorff measure which is clear from the context.

$BV(\Omega)$ is the class of locally summable functions in Ω whose first generalized derivatives are measures.

[1]An expanded exposition with some generalizations will appear in the “St. Petersburg Mathematical Journal” (added in proofs).

Denote by $\chi(E)$ the characteristic function of E; by $P_\Omega(E)$ we mean the perimeter of the set $E \subset \Omega$; i.e., $P(E) = \|\text{grad}\chi_E\|_{BV(\Omega)}$. The reader can find detailed explanations in [2], Chapter 6.

We will use the Fleming-Rishel formula [4]

$$\|f\|_{BV(\Omega)} = \int_{-\infty}^{\infty} P_\Omega(E_t)\, dt, \tag{1.1}$$

where $f \in BV(\Omega)$, $E_t = \{x \mid f(x) \geq t\}$, and also a related formula

$$\nabla f(E) = \int_{-\infty}^{+\infty} \nabla\chi_{E_t}\, dt, \tag{1.2}$$

where E is any measurable subset of Ω, see, for instance, Theorem 14 in [2] or Lemma 6.6.5(1) in [5].

2. One-sided densities

Recall that almost all points of $\partial\Omega$ are regular. Denote by $\nu(x)$ the unit normal vector to T_x at a regular point x and let $B_x^\nu(r) = B_x(r) \cap \{y \mid y\nu \geq 0\}$. Denote by $\Theta_E^\nu(x)$ the limit

$$\lim_{r\to 0} 2v_n^{-1} r^{-n} H_n(B_x^\nu(r) \cap E).$$

We call it the one-sided density of the set E at $x \in \partial\Omega$. Upper and lower one-sided densities $\overline{\Theta}_E^\nu(x)$, $\underline{\Theta}_E^\nu(x)$ are defined similarly. At any regular point there are two unit normals and, correspondingly, two one-sided densities.

Remark 2.1. It is easy to see that if $\Theta_G^\nu(x) = 1$, then

$$\Theta_E^\nu(x) = \lim_{r\to 0} \frac{H_n(B_x(r) \cap G \cap E)}{H_n(B_x(r) \cap G)}. \tag{2.1}$$

The following statement is a simple consequence of the isoperimetric inequality for a ball.

Proposition 2.2 (Isoperimetric inequality). *Let E be a set with $P(E) < \infty$ and Q a sector, $Q = \{x \in \mathbb{R}^n \mid \sum x_i^2 < 1,\ a < x_n < 1\}$, where $a \leq 1/2$. Then the isoperimetric inequality*

$$\min\{H_n(E \cap Q),\ H_n((\mathbb{R}^n \setminus E) \cap Q)\} \leq c_n P_Q(E)^{\frac{n}{n-1}}, \tag{2.2}$$

holds, and the constant $c_n > 0$ depends only on dimension.

Lemma 2.3. *For any regular point $x \in \partial\Omega$ and any normal $\nu(x)$ either $\Theta_\Omega^\nu(x) = 1$ or $\Theta_{\mathbb{R}^n\setminus\Omega}^\nu(x) = 1$.*

Note that for normals ν, $-\nu$ all pairwise combinations are possible even for a set of points of positive Hausdorff measure μ.

Proof. For a regular point $x \in \partial\Omega$ and a normal ν at x, consider sets $B_i^\nu = B_x^\nu(r_i)$, where $r_i \to +0$ as $i \to \infty$. The definition of $T_x = \mathrm{Tan}_x^{n-1}(\partial\Omega)$ implies that for any $\epsilon > 0$ and sufficiently large i the inequality $\mu(A_i \cap \partial\Omega) < \epsilon r_i^{n-1}$ holds. Here $A_i = B_i^\nu \setminus C_{\epsilon i}$ and $C_{\epsilon i}$ is the ϵr-neighborhood of T_x. Lemma follows immediately from this fact and the isoperimetric inequality (2.2). □

Example 2.4. Consider a sequence of small bubbles (disjoint round balls) $B_{x_i}(r_i)$ located in a unit closed ball $\bar{B}_0(1)$. We choose these bubbles in such a way, that every point $p \in S_0(1)$ is the limit of some subsequence of the bubbles and, besides, there is no other limit points. In addition, suppose that the radii of these balls vanish very fast. Consider the set $\Omega = \cup B_{x_i}(r_i)$. This set is disconnected, but in dimensions greater than 2 we can connect the bubbles by thin tubules. Then Ω becomes a region. It is easy to see that $S_0(1) \cap \partial^*\Omega$ is empty. Moreover, at almost every point x of $S_0(1)$ the approximative tangent plane exists (so x is a regular point) but $\Theta_\Omega^\nu(x) = 0$ for both normals at x.

Denote by Γ the set of all regular points $x \in \partial\Omega$ such that $\Theta_\Omega^\nu(x) = 1$ for at least one normal ν. It is not difficult to see that this set contains the reduced boundary $\partial^*\Omega$. Indeed, $\Theta_\Omega^{-\nu_F}(x) = 1$, where ν_F is the normal in the sense of Federer.

Remark 2.5. It is well known that perimeter $P(\Omega) = \mu(\partial^*\Omega)$. Recall that if $P(\Omega) < \infty$, then $\mathrm{var}\nabla\chi_\Omega(\partial\Omega \setminus \partial^*\Omega) = 0$ and

$$\nabla\chi_\Omega(E) = -\int_E \nu_F(x)\,\mu(dx) \tag{2.3}$$

for any measurable set $E \subset \partial^*\Omega$, see, for instance, Theorem 6.2.2(1) in [2].

Lemma 2.6. *Any μ-rectifiable set A can be equipped with a measurable field ν of* (*unit*) *normals.*

Proof. Up to a set of measure zero, A consists of regular points. Almost all these points are contained in the union of a countable set of $(n-1)$-dimensional C^1-smooth manifolds M_i. Note that up to a set of measure zero each point $x \in A$ belongs to only one M_i. For $x \in A \cap M_i$ the approximative tangent plane to A at x coincides with the tangent plane T_xM_i. The sets $A \cap M_i$ are measurable. We can orient each manifold M_i by a continuous field of normals. Now for $x \in A \cap M_i$, let us take the normal to M_i at x in the capacity of $\nu(x)$. This defines a measurable field of normals. □

Remark 2.7. It is obvious that a measurable vector field of normals obtained in Lemma 2.6 is not unique, there are infinity many of them. In particular, it is possible to choose vector field ν in such a way that at each point $x \in \partial^*\Omega$ the normal $-\nu(x)$ is the normal to Ω in the sense of Federer. Let us fix such a vector field and call it *standard field*. Thus, the standard field is not only measurable but is located on a fixed countable set of C^1-smooth surfaces M_i, besides it is continuous along every such surface, and is opposite to the normal in the sense of Federer at points $x \in \partial^*\Omega$.

Lemma 2.8. *Let ν be a measurable field of normals along $\Gamma \subset \partial\Omega$, and E be a measurable subset of Ω. Then the set of $x \in \partial\Omega$ such that $\Theta^\nu_E(x) = 1$ is measurable.*

Proof. Recall that $\partial\Omega$ is supposed to be μ-rectifiable. First assume that the field ν is standard. Let $\{M_i\}$ be a set of surfaces chosen as above, in Remark 2.7 . The sets $M_i \cap \partial\Omega$ are measurable. As before, let $B^\nu_x(r) = B_x(r) \cap \{y \mid y\nu \geq 0\}$. The functions $\phi^r_i(x) = 2v_n^{-1} r^{-n} H_n(B^\nu_x(r) \cap E)$ defined on $M_i \cap \partial\Omega$ are continuous, therefore they are measurable. Let us extend these functions by zero to the whole $\partial\Omega$. The function $\phi^r = \sum_i \phi^r_i$ (defined on $\partial\Omega$) is measurable too, therefore the function $\phi(x) = \liminf_{r\to 0} \phi^r(x)$ is measurable. Hence, the set $\{x \in \partial\Omega \mid \Theta^\nu_E(x) = 1\}$ $= \{x \in \partial\Omega \mid \phi(x) = 1\}$ is measurable. The same is correct for the field $-\nu$. Now let $\tilde{\nu}$ be any measurable unit vector field on $\partial\Omega$. Then the sets $\{\nu = \tilde{\nu}\}$ and $\{-\nu = \tilde{\nu}\}$ are measurable. Thus, the set $\{\Theta^{\tilde{\nu}}_E(x) = 1\} \subset \partial\Omega$ is measurable as well. □

For a H_n-measurable set E and a normal ν to Γ at x, denote

$$\partial^\nu_\Gamma E = \{x \in \partial\Omega \mid \Theta^\nu_E(x) = 1\},$$

$$\partial^1_\Gamma E = (\partial^\nu_\Gamma E) \cup (\partial^{-\nu}_\Gamma E), \qquad \partial^2_\Gamma E = (\partial^\nu_\Gamma E) \cap (\partial^{-\nu}_\Gamma E). \tag{2.4}$$

It is easy to see that for standard normal field ν (see Remark 2.7)

$$\partial^1_\Gamma E = \partial^\nu_\Gamma E, \qquad \partial^2_\Gamma E = \partial^{-\nu}_\Gamma E. \tag{2.5}$$

Roughly speaking, $\partial^1_\Gamma E$ is the set where E “approaches” to Γ with density 1 at least from one side, and $\partial^2_\Gamma E$ is the part of Γ, where E “approaches” to Γ with density 1 from both sides.

Lemma 2.9. *Let $E \subset \Omega$ and $P(E) < \infty$. Then at almost all points $x \in \Gamma$ the one-sided density $\Theta^\nu_E(x)$ is either zero or one.*

Proof. It suffices to prove the lemma only for the case of a standard normal field and for almost all points $x \in \Gamma$ only (see Lemma 2.6 and Remark 2.7). Let $\{M_i\}$ be a family of C^1-smooth manifolds used for the definition of standard fields of normals. Let M be one of these manifolds M_i, $x \in M$. Choose a region Ω' such that some neighborhood of x on M is a part of the boundary Ω' and $-\nu(x)$ is the normal in the sense of Federer to Ω' at x. (For instance, one can pick a point $p \in M$ and take adjusting to p component of $B_p(\rho) \setminus M$ for a small $\rho > 0$ in the capacity of Ω'.) Consider the region $\Omega' \cap \Omega = \widetilde{\Omega}$. It is easy to see that $-\nu(x)$ is Federer's normal to $\widetilde{\Omega}$ at points $x \in \Gamma \cap M$. Besides $\Theta^\nu_{\widetilde{\Omega}}(x) = 1$. Furthermore, at points $x \in \Gamma \cap M$

$$\lim_{r\to 0} 2v_n^{-1} r^{-n} \mathrm{Vol}((B_x(r) \cap \widetilde{\Omega}) \triangle B^\nu_x(r)) = 0.$$

Lemma 13 from [2] implies that the trace of the function $\chi_{E\cap\widetilde{\Omega}}$ on the reduced boundary $\partial^*\widetilde{\Omega}$ is equal to $\chi_{(\partial^* E)\cap(\partial^*\widetilde{\Omega})}(x)$ for almost all $x \in \partial^*\widetilde{\Omega}$. By definition, the trace of $\chi_{E\cap\widetilde{\Omega}}$ on the reduced boundary $\partial^*\widetilde{\Omega}$ is equal to

$$\lim_{r\to 0}\frac{\int_{B_x(r)}\chi_{E\cap\widetilde{\Omega}}\,dx}{\mathrm{Vol}(B_x(r)\cap\widetilde{\Omega})}=\lim_{r\to 0}\frac{\mathrm{Vol}(B_x(r)\cap\widetilde{\Omega}\cap E)}{\mathrm{Vol}(B_x(r)\cap\widetilde{\Omega})}.$$

If this limit is zero, then, since $\Theta^{\nu}_{\widetilde{\Omega}}(x)=1$, we obtain $\Theta^{\nu}_{E}(x)=0$. Similarly, if the limit is one, then $\Theta^{\nu}_{E}(x)=1$. □

Now it is clear that for any set E with finite perimeter the reduced boundary

$$\partial^* E=(\partial^1_\Gamma E)\setminus(\partial^2_\Gamma E). \tag{2.6}$$

Remark 2.10. Note that two last lemmas hold true for any rectifiable set $\Gamma\subset\mathbb{R}^n$

Corollary 2.11. *Suppose that G_1, G_2 are such that $\Theta^{\nu}_{G_1}(x)=\Theta^{\nu}_{G_2}(x)=1$. Let family of sets $\mathcal{B}^{\nu}_x(r)$ be such that*

$$B_x(\rho_1(r))\cap G_1\subset\mathcal{B}^{\nu}_x(r)\subset B_x(\rho_2(r))\cap G_2, \tag{2.7}$$

where $\rho_2(r)\to 0$ as $r\to 0$. Then for any H_n-measurable set $E\subset\mathbb{R}^n$ with finite perimeter it follows

$$\Theta^{\nu}_{E}(x)=\lim_{r\to 0}\frac{H_n(\mathcal{B}^{\nu}_x(r)\cap E)}{H_n(\mathcal{B}^{\nu}_x(r))}. \tag{2.8}$$

3. Boundary trace

Denote $E_t=\{x\in\Omega\mid f(x)\geq t\}$. For a function $f\in BV(\Omega)$, we define boundary trace[2] $f^{\nu}(x):\partial^{\nu}_{\Gamma}\Omega\to\mathbb{R}$ with respect to a normal vector field ν by the equation $f^{\nu}(x)=\sup\{t\mid x\in\partial^{\nu}_{\Gamma}E_t\}$.

In the case $x\in\partial^2_\Gamma\Omega$, we define also upper and lower traces by the equalities

$$f^*(x)=\max\{f^{\nu}(x),f^{-\nu}(x)\},\ \ f_*(x)=\min\{f^{\nu}(x),f^{-\nu}(x)\}.$$

In the case $x\in\partial^*\Omega$, we put $f^*(x)=f^{\nu}(x)$, where $-\nu$ is the normal in the sense of Federer. In this case we do not define $f_*(x)$ at all.[3]

It is clear that $f^*(x)=\sup\{t\mid x\in\partial^1_\Gamma E_t\}$, $f_*(x)=\sup\{t\mid x\in\partial^2_\Gamma E_t\}$.

Lemma 3.1. *Let $\partial\Omega$ be a rectifiable set and $f\in BV(\Omega)$. Then f^{ν} is measurable and*

$$\mu(\{x\in\partial^{\nu}_{\Gamma}\Omega\mid f^{\nu}(x)\geq t\})=\mu(\partial^{\nu}_{\Gamma}E_t) \tag{3.1}$$

for almost all $t\in\mathbb{R}$.

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[2]Here our terminology is different from one in [2], [5]. Namely, we use terms trace and average trace instead of rough trace and trace, correspondingly.

[3]However, if one expends f to all $\mathbb{R}^n$, it is reasonable to consider all $\partial\Omega$ like $\partial^1_\Gamma\Omega$ and so to have upper and lower traces everywhere on Γ.

Remark 3.2. 1) Similarly to Lemma 3.1, one can prove that f^* and f_* are measurable and

$$\mu(\{x \in \partial^1_\Gamma \Omega \mid f^*(x) \geq t\}) = \mu(\partial^1_\Gamma E_t), \tag{3.2}$$

$$\mu(\{x \in \partial^2_\Gamma \Omega \mid f_*(x) \geq t\}) = \mu(\partial^2_\Gamma E_t). \tag{3.3}$$

2) In fact, instead of (3.1) we will prove that

$$\mu\Big(\{x \in \partial^\nu_\Gamma \Omega \mid f^\nu(x) \geq t\} \Delta \partial^\nu_\Gamma E_t\Big) = 0$$

for all $t \in \mathbb{R}$ except countable subset and so on.

Proof. Denote $B_t = \{x \in \partial^\nu_\Gamma \Omega \mid f^\nu(x) \geq t\}$, $Y_t = \partial^\nu_\Gamma E_t$, and $X_t = B_t \setminus Y_t$. One can see that $Y_t \subset B_t$. Thus, it remains to prove that $\mu(X_t) = 0$.

The sets Y_t are measurable and the sets X_t are disjoint. It is clear that the inclusions $Y_{t_0} \supset Y_{t_1}$ and $Y_{t_0} \cup X_{t_0} \supset Y_{t_1} \cup X_{t_1}$ hold for $t_0 < t_1$. The latter implies $Y_{t_0} \supset X_{t_1}$. Thus

$$\Big(\bigcap_{t<t_1} Y_t\Big) \setminus Y_{t_1} \supset X_{t_1}.$$

From the other hand, the sets $(\bigcap_{t<t_1} Y_t) \setminus Y_{t_1}$ are measurable and disjoint. Thus $\mu\Big((\bigcap_{t<t_1} Y_t) \setminus Y_{t_1}\Big) = 0$ for almost all $t_1 \in \mathbb{R}$. This implies that the sets X_t (being subsets of measure zero sets) are measurable and of measure zero for almost all $t \in \mathbb{R}$. Thereby the sets B_t are measurable. □

Lemma 3.3. *For any $f \in BV(\Omega)$ and almost all $x \in \Gamma$*

$$-f^\nu(x) = (-f)^\nu(x). \tag{3.4}$$

Proof. Lemma 3.3 is equivalent to the fact that for almost all $x \in \Gamma$

$$\sup\{t \mid x \in \partial^\nu_\Gamma E_t\} = \inf\{t \mid x \in \partial^\nu_\Gamma (\Omega \setminus E_t)\}.$$

This equality means that

$$\sup\{t \mid \Theta^\nu_{E_t}(x) = 1\} = \inf\{t \mid \Theta^\nu_{(\Omega\setminus E_t)}(x) = 1\}.$$

In its turn, this equivalent to the equality

$$\sup\{t \mid \Theta^\nu_{E_t}(x) = 1\} = \inf\{t \mid \Theta^\nu_{E_t}(x) = 0\}.$$

Denote by L, R the left and the right terms of the last equality. It is easy to see that $\Theta^\nu_{E_t}(x)$ is a nonincreasing function on t. So $L \leq R$. Consider the set of points x such that $L(x) < R(x)$. It suffices to prove that μ-measure of this set is zero. Let $\{t_i\}_{i=1}^\infty$ be a countable dense in $\mathbb{R}$ set such that $P(E_{t_i}) < \infty$. If $L(x) < R(x)$, then there exists t_i such that $L(x) < t_i < R(x)$. Now our statement follows from the equality $\mu\{x \mid 0 < \Theta^\nu_{E_t}(x) < 1\} = 0$.

□

Corollary 3.4. *For any $f \in BV(\Omega)$ and for almost all $x \in \Gamma$*

$$(f^\nu)^+ = (f^+)^\nu, \qquad (f^\nu)^- = (f^-)^\nu. \tag{3.5}$$

Proof. The first equality easy follows from definitions. The second one follows from Lemma 3.3. Indeed, $(f^-)^\nu = ((-f)^+)^\nu = ((-f)^\nu)^+ = (-f^\nu)^+ = (f^\nu)^-$. □

Lemma 3.5. *For every $f, g \in BV(\Omega)$ and for almost all $x \in \Gamma$*

$$(f+g)^\nu(x) = f^\nu(x) + g^\nu(x). \tag{3.6}$$

Proof. First we show that $(f+g)^\nu(x) \geq f^\nu(x) + g^\nu(x)$ for all $x \in \Gamma$. Indeed, let numbers $F < f^\nu(x)$ and $G < g^\nu(x)$ be such that the sets $E^f_F = \{x \mid f(x) > F\}$ and $E^g_G = \{x \mid g(x) > G\}$ have finite perimeters. Then $\Theta^\nu_{E^f_F}(x) = 1$ and $\Theta^\nu_{E^g_G}(x) = 1$.

Now let $A = E^{f+g}_{F+G}$. Then

$$A = \{x \mid f(x) + g(x) > F + G\} \supset E^f_F \cap E^g_G.$$

Therefore $\Theta^\nu_A(x) = 1$ and so

$$(f+g)^\nu(x) = \sup\{t \mid \Theta^\nu_{E^{f+g}_t} = 1\} \geq F + G.$$

Taking the limit as F tends to $f^\nu(x)$ and G tends to $g^\nu(x)$, we get

$$(f+g)^\nu(x) \geq f^\nu(x) + g^\nu(x).$$

Now one can obtain the opposite inequality for almost all $x \in \Gamma$ by applying Lemma 3.3. Indeed, for almost all $x \in \Gamma$

$$\begin{aligned} -(f+g)^\nu(x) &= ((-f) + (-g))^\nu(x) \geq (-f)^\nu(x) + (-g)^\nu(x) \\ &= -f^\nu(x) - g^\nu(x). \end{aligned}$$

□

Lemma 3.6. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function. If $f, \phi(f) \in BV(\Omega)$, then for almost all $x \in \Gamma$*

$$(\phi(f))^\nu(x) = \phi(f^\nu(x)). \tag{3.7}$$

Proof. The lemma easily follows from the equality

$$\{x \in \Omega \mid \phi(f(x)) \geq \phi(t)\} = \{x \in \Omega \mid f(x) \geq t\}.$$

□

Lemma 3.7. *If $f, g, fg \in BV(\Omega)$, then for almost all $x \in \Gamma$ we have*

$$(fg)^\nu(x) = f^\nu(x) g^\nu(x). \tag{3.8}$$

Proof. It suffices to prove (3.8) only for the case $f, g \geq 1$. It easily follows from Lemma 3.3, its corollary, and the equality $f = (f^+ + 1) - (f^- + 1)$. By Lemma 3.5, we have now:

$$\begin{aligned} (fg)^\nu = \left(e^{\ln(fg)}\right)^\nu &= e^{(\ln f + \ln g)^\nu} = e^{(\ln f)^\nu + (\ln g)^\nu} \\ &= e^{\ln(f^\nu) + \ln(g^\nu)} = f^\nu g^\nu. \end{aligned}$$

□

4. An integral formula for norm of trace

Definition 4.1. Let us define

$$||f||_\Gamma = \int_{\partial^*\Omega} |f^*|\, d\mu + \int_{\partial^2_\Gamma\Omega} (f^* - f_*)\, d\mu. \tag{4.1}$$

For $||f||_\Gamma < \infty$ we say that f is *trace-summable.*

Lemma 4.2.

$$||f||_\Gamma = ||f^+||_\Gamma + ||f^-||_\Gamma. \tag{4.2}$$

Proof. We have

$$\begin{aligned} f^* - f_* &= |f^\nu - f^{-\nu}| = |((f)^+)^\nu - ((f)^-)^\nu - ((f)^+)^{-\nu} + ((f)^-)^{-\nu}| \\ &= |(f^+)^\nu - (f^+)^{-\nu}| + |(f^-)^{-\nu} - (f^-)^{-\nu}| \\ &= ((f^+)^* - (f^+)_*) + ((f^-)^* - (f^-)_*). \end{aligned}$$ □

Lemma 4.3. *Suppose a function $f \in BV(\Omega)$ to be nonnegative and trace-summable. Let a function $\eta\colon \Gamma \to \mathbb{R}^k$, $k \geq 1$, be measurable and bounded. Then*

$$\int_0^{+\infty} \int_{\Gamma\cap\partial^* E_t} \eta\, d\mu\, dt = \int_{\partial^*\Omega} f^*\eta\, d\mu + \int_{\partial^2_\Gamma\Omega} (f^* - f_*)\eta\, d\mu. \tag{4.3}$$

Obviously, we can assume $k = 1$. We omit the proof which is based on the equation (2.6) and Lemma 3.1.

Substituting 1 for η, we obtain the following corollary.

Corollary 4.4. *If a nonnegative function $f \in BV(\Omega)$ is trace-summable, then*

$$\int_0^{+\infty} \mu(\Gamma \cap \partial^* E_t)\, dt = ||f||_\Gamma. \tag{4.4}$$

In addition, f has the summable trace if and only if the left part of (4.4) *is finite.*

5. Summability of traces

Theorem 4 in [2] (or, that is the same, Theorem 6.5.2(1) in [5]) can be generalized to our case in somewhat different forms. Here we give one of such generalizations.

Theorem 5.1. *Let $\partial\Omega$ be a rectifiable set. Then the inequality*

$$\inf_c \{||f - c||_\Gamma\} \leq k\|f\|_{BV(\Omega)}, \tag{5.1}$$

where k does not depend on f, holds for every $f \in BV(\Omega)$ if and only if for every set $E \subset \Omega$ with finite perimeter

$$\min\left\{\mu(\Gamma \cap \partial^* E),\ \mu(\Gamma \cap \partial^*(\Omega \setminus E)\right\} \leq kP_\Omega(E). \tag{5.2}$$

The proof basically follows the same way as Theorem 6.5.2/1 in [5].

Theorem 5.1 gives a global criterion for summability of traces. One can obtain local criteria by using partition of unity, compare [2], [5].

6. Extension of a function in $BV(\Omega)$ to $\mathbb{R}^n$ by a constant

In this section we assume that $P(\Omega) < \infty$ and $\partial\Omega$ is a rectifiable set.

For a function $f\colon \Omega \to \mathbb{R}$, denote by f_c the function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by equalities $f_c(x) = f(x)$ for $x \in \Omega$ and $f_c(x) = c$ for $x \notin \Omega$, where c is a constant.

Lemma 6.1. *The following equality holds*

$$\|f_c\|_{BV(\mathbb{R}^n)} = \|f\|_{BV(\Omega)} + ||f - c||_\Gamma. \tag{6.1}$$

Proof. Without loss of generality we can assume that $c = 0$; indeed, we can consider $f - c$ instead of f. Formula (4.2) allows to assume that $f \geq 0$. As usual, $E_t = \{x \in \Omega \mid f_0 > t\}$. By (1.1) and (4.4) we have

$$\|f_0\|_{BV(\mathbb{R}^n)} = \int_0^{+\infty} P(\{x \in \mathbb{R}^n \mid f_0 > t\})\, dt$$
$$= \int_0^{+\infty} \Big(P_\Omega(E_t) + \mu(\Gamma \cap \partial^* E_t)\Big)\, dt = \|f\|_{BV(\Omega)} + ||f||_\Gamma. \qquad \square$$

One can ask if it is possible to enlarge Ω by removing $\partial^2_\Gamma\Omega$ and thus to reduce our considerations to the case when normals in the sense of Federer exist almost everywhere on $\partial\Omega$. Sometimes it is possible. For instance, let $\Omega = D^2 \setminus \cup_{i=1}^\infty I_i$ be the disk with a sequence of intervals removed in such a way, that the sum of lengths of I_i is finite. Then for every $f \in BV(\Omega)$ such that

$$\int_{\cup_{i=1}^\infty I_i} (f^* - f_*) < \infty$$

one can extend f to a function $\tilde f \in BV(D^2)$. Unfortunately a slightly more complicate example shows that this is not always possible.

Example 6.2. Let $K \subset [0,1]$ be a Kantor set of positive length. Define a region Ω by the equality

$$\Omega = B_{(0,0)}(2) \setminus \{(x,y) \mid x \in [0,1],\ |y| \leq (\operatorname{dist}(x,K))^2\}. \tag{6.2}$$

One can easily verify that for all points of $K \times \{0\}$ the both one-sided densities are equal to one and so these points form $\partial^2_\Gamma\Omega$. However it is impossible to enlarge Ω by removing $K \times \{0\}$.

7. Embedding theorems

In this and the next section we assume functions of $BV(\Omega)$ to be extended by zero outside Ω and so we consider the open set $\Omega \cup (\mathbb{R}^n \setminus \operatorname{Cl}(\Omega))$ instead of Ω. This set is not connected but this does not play any role for our considerations.

Theorem 7.1. *Let $\partial\Omega$ be a μ-rectifiable set. Then the inequality*

$$\left[\int_\Omega f^{\frac{n}{n-1}}\,dx\right]^{\frac{n-1}{n}} \le n c_n^{-\frac{1}{n}} \left\{\|f\|_{BV(\Omega)} + \|f\|_\Gamma\right\} \tag{7.1}$$

holds for every function $f \in BV(\Omega)$, and the constant $nc_n^{-\frac{1}{n}}$ is exact.

The theorem can be proved in the same way as Theorem 6.5.7 in [5].

8. The Gauss-Green formula

Theorem 8.1 (The Gauss-Green formula). *Let Γ be a μ-rectifiable set. Assume that $\partial\Omega$ is equipped with a standard field of unit normals ν. Let $f \in BV(\Omega)$ be trace-summable. Then*

$$\nabla f(\Omega) = \int_{\partial^*\Omega} f^\nu(x)\nu(x)\,d\mu(x) + \int_{\partial^2_\Gamma\Omega} \big(f^\nu(x) - f^{-\nu}(x)\big)\nu(x)\,d\mu(x). \tag{8.1}$$

Proof. It suffices to prove (8.1) for nonnegative functions f. Indeed, to prove the theorem in the general case, it suffices to apply (8.1) to f^+ and f^- and then to use Corollary 3.4.

It is clear that the right side of (8.1) does not depend on choice of the normal vector field ν. Note that if $f^*(x) \ne f_*(x)$, then the normal in the sense of Federer to E_t at x does exist for all $t \in (f_*(x),\ f^*(x))$ and does not depend on t. Therefore we can suppose that at each such point x the normal $-\nu(x)$ coincides with the normal in the sense of Federer to E_t, $f_*(x) < t < f^*(x)$. For such a normal field ν formula (8.1) can be written in the following way:

$$\nabla f(\Omega) = \int_{\partial^*\Omega} f^*(x)\nu(x)\,d\mu(x) + \int_{\partial^2_\Gamma\Omega} \big(f_*(x) - f^*(x)\big)\nu(x)\,d\mu(x). \tag{8.2}$$

If $P(E) < \infty$, then obviously $\nabla\chi_E(\mathbb{R}^n) = 0$. Applying (1.2) to the left term of 8.2, we get

$$\nabla f(\Omega) = \int_0^\infty \nabla\chi_{E_t}(\Omega)\,dt = -\int_0^\infty \nabla\chi_{E_t}(\mathbb{R}^n \setminus \Omega)\,dt = -\int_0^\infty \nabla\chi_{E_t}(\Gamma \cap \partial^* E_t)\,dt.$$

On the other hand, by (2.3) we have

$$\nabla\chi_{E_t}(\Gamma \cap \partial^* E_t) = -\int_{\Gamma\cap\partial^* E_t} \nu_{E_t}(x)\,d\mu(x) = -\int_{\Gamma\cap\partial^* E_t} \nu(x)\,d\mu(x),$$

where ν_{E_t} is the normal in the sense of Federer to E_t. The latter equality holds since $\nu_{E_t}(x) = \nu(x)$ for almost all $x \in \Gamma \cap \partial^* E_t$ and $\mu(E_t \setminus \cup_{\tau>t} E_\tau) = 0$ for almost all $t \in \mathbb{R}$.

Therefore, by (4.3) for $\eta = \nu$ we obtain

$$\begin{aligned}\nabla f(\Omega) &= -\int_0^{+\infty} \nabla \chi_{E_t}(\Gamma \cap \partial^* E_t)\, dt \\ &= \int_0^{+\infty} \int_{\Gamma \cap \partial^* E_t} \nu(x)\, d\mu(x) \\ &= \int_\Gamma f^*(x)\nu(x)\, d\mu(x) + \int_{\partial^2_\Gamma \Omega} (f^*(x) - f_*(x))\nu(x)\, d\mu(x).\end{aligned}$$

The theorem is proved. □

9. Average trace of $BV(\Omega)$ functions

Let Ω be a region with the rectifiable boundary $\partial\Omega$. Suppose that a function $f \in BV(\Omega)$ is summable in a neighborhood of some point $x \in \Gamma$. Define upper and lower average traces of f at x with respect to a normal ν by the equalities

$$\overline{f}(x,\nu) = \limsup_{r \to 0}\, 2v_n^{-1} r^{-n} \int_{B_r^\nu(x)} f(y)\, dy,$$

$$\underline{f}(x,\nu) = \liminf_{r \to 0}\, 2v_n^{-1} r^{-n} \int_{B_r^\nu(x)} f(x)\, dy.$$

If $\overline{f}(x,\nu) = \underline{f}(x,\nu)$, then their common value is called *average trace*. Denote it by $\widetilde{f}(x,\nu)$. First let us prove some properties of average trace of nonnegative functions.

Lemma 9.1. *Suppose that $f \in BV(\Omega)$ is a nonnegative and local summable function. Then $\underline{f}(x,\nu) \geq f^\nu(x)$.*

The proof of this lemma basically is similar to Lemma 6.6.2 in [5].

Theorem 9.2. *Let a function $f \in BV(\Omega)$ be bounded. Then the average trace $\tilde{f}(x,\nu)$ of the function f exists almost everywhere on Γ and coincides with the trace $f^\nu(x)$.*

Proof. Let $|f| < C$. From Lemma 3.6 and Lemma 9.1 it follows that

$$f^\nu(x) = (f+C)^\nu(x) + (-C)^\nu(x) \leq \underline{(f+C)}(x,\nu) - C = \underline{f}(x,\nu).$$

Applying this to $-f$ we have

$$(-f)^\nu(x) \leq \underline{(-f)}(x,\nu).$$

Therefore by Lemma 3.3 we obtain that at almost all $x \in \Gamma$

$$f^\nu(x) \geq \overline{f}(x,\nu).$$

The theorem is proved. □

The theorem is valid for unbounded functions too, but the proof becomes more complicated and we omit it here.

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Dirac Equation as a Special Case of Cosserat Elasticity

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We dedicate our paper to Vladimir Maz'ya whose works set the standard for applicable rigorous mathematical analysis

Abstract. We suggest an alternative mathematical model for the electron in which the dynamical variables are a coframe (field of orthonormal bases) and a density. The electron mass and external electromagnetic field are incorporated into our model by means of a Kaluza–Klein extension. Our Lagrangian density is proportional to axial torsion squared. The advantage of our approach is that it does not require the use of spinors, Pauli matrices or covariant differentiation. The only geometric concepts we use are those of a metric, differential form, wedge product and exterior derivative. We prove that in the special case with no dependence on the third spatial coordinate our model is equivalent to the Dirac equation. The crucial element of the proof is the observation that our Lagrangian admits a factorisation.

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1. Introduction

The Dirac equation is a system of four homogeneous linear complex partial differential equations for four complex unknowns. The unknowns (components of a bispinor) are functions of time and the three spatial coordinates. The Dirac equation is the accepted mathematical model for an electron and its antiparticle, the positron, in a given external electromagnetic field. One of the main applications of the Dirac equation is spectral-theoretic: it determines with high accuracy the energy levels of the hydrogen atom.

The geometric interpretation of the Dirac equation is rather complicated. It relies on the use of notions such as

- spinor,
- Pauli matrices,

- covariant derivative (note that formula (2.1) for the covariant derivative of a spinor field is quite tricky).

There is also a logical problem with the Dirac equation in that distinguishing the electron from the positron forces one to resort to the concept of negative energy. Finally, the electromagnetic field is incorporated into the Dirac equation by means of a formal substitution which does not admit a simple geometric interpretation.

The purpose of this paper is to formulate an alternative mathematical model for the electron and positron, a model which is geometrically much simpler. The advantage of our approach is that it does not require the use of spinors, Pauli matrices or covariant differentiation. The only geometric concepts we use are those of a

- metric,
- differential form,
- wedge product,
- exterior derivative.

Our model overcomes the logical problem of distinguishing the electron from the positron: these correspond to clockwise and anticlockwise rotations of the coframe. And the electromagnetic field is incorporated into our model by means of a Kaluza–Klein extension which has a simple geometric interpretation.

The paper has the following structure. In Section 2 we introduce our notation and in Section 3 we formulate the Dirac equation. In Section 4 we formulate our mathematical model and in Section 5 we translate our model into the language of bispinors. In Section 6 we prove Theorem 6.2 which is the main result of the paper: this theorem establishes that in the special case with no dependence on x^3 our mathematical model is equivalent to the Dirac equation. The crucial element of the proof of Theorem 6.2 is the observation that our Lagrangian admits a factorisation; this factorisation is the subject of Lemma 6.1. The concluding discussion is contained in Section 7.

2. Notation and conventions

Throughout this paper we work on a 4-manifold M equipped with prescribed Lorentzian metric g. All constructions presented in the paper are local so we do not make a priori assumptions on the geometric structure of spacetime $\{M, g\}$. The metric g is not necessarily the Minkowski metric.

Our notation follows [1, 2]. In particular, in line with the traditions of particle physics, we use Greek letters to denote tensor (holonomic) indices.

By ∇ we denote the covariant derivative with respect to the Levi-Civita connection. It acts on a vector field and a spinor field as $\nabla_\alpha v^\beta := \partial_\alpha v^\beta + \Gamma^\beta{}_{\alpha\gamma} v^\gamma$ and

$$\nabla_\alpha \xi^a := \partial_\alpha \xi^a + \frac{1}{4}\sigma_\beta{}^{a\dot{c}}(\partial_\alpha \sigma^\beta{}_{b\dot{c}} + \Gamma^\beta{}_{\alpha\gamma}\sigma^\gamma{}_{b\dot{c}})\xi^b \tag{2.1}$$

respectively, where $\Gamma^{\beta}{}_{\alpha\gamma} = \left\{ {\beta \atop \alpha\gamma} \right\} := \frac{1}{2} g^{\beta\delta}(\partial_\alpha g_{\gamma\delta} + \partial_\gamma g_{\alpha\delta} - \partial_\delta g_{\alpha\gamma})$ are the Christoffel symbols and σ_β are Pauli matrices.

We identify differential forms with covariant antisymmetric tensors. Given a pair of real covariant antisymmetric tensors P and Q of rank r we define their dot product as $P \cdot Q := \frac{1}{r!} P_{\alpha_1 \ldots \alpha_r} Q_{\beta_1 \ldots \beta_r} g^{\alpha_1 \beta_1} \ldots g^{\alpha_r \beta_r}$. We also define $\|P\|^2 := P \cdot P$.

3. The Dirac equation

The following system of linear partial differential equations on M is known as the *Dirac equation*:

$$\sigma^{\alpha a \dot b}(i\nabla + A)_\alpha \eta_{\dot b} = m\xi^a, \qquad \sigma^{\alpha}{}_{a\dot b}(i\nabla + A)_\alpha \xi^a = m\eta_{\dot b}\,. \tag{3.1}$$

Here ξ^a, $\eta_{\dot b}$ is a bispinor field which plays the role of dynamical variable (unknown quantity), m is the electron mass and A is the prescribed electromagnetic covector potential. The corresponding Lagrangian density is

$$\begin{aligned} L_{\mathrm{Dir}}(\xi,\eta) := \Big[&\frac{i}{2}(\bar\xi^{\dot b} \sigma^{\alpha}{}_{a\dot b} \nabla_\alpha \xi^a - \xi^a \sigma^{\alpha}{}_{a\dot b} \nabla_\alpha \bar\xi^{\dot b} + \bar\eta_a \sigma^{\alpha a \dot b} \nabla_\alpha \eta_{\dot b} - \eta_{\dot b} \sigma^{\alpha a \dot b} \nabla_\alpha \bar\eta_a) \\ &+ A_\alpha(\xi^a \sigma^{\alpha}{}_{a\dot b} \bar\xi^{\dot b} + \bar\eta_a \sigma^{\alpha a \dot b} \eta_{\dot b}) - m(\xi^a \bar\eta_a + \bar\xi^{\dot b} \eta_{\dot b}) \Big] \sqrt{|\det g|}\,. \end{aligned} \tag{3.2}$$

4. Our model

A *coframe* ϑ is a quartet of real covector fields ϑ^j, $j = 0, 1, 2, 3$, satisfying the constraint

$$g = \vartheta^0 \otimes \vartheta^0 - \vartheta^1 \otimes \vartheta^1 - \vartheta^2 \otimes \vartheta^2 - \vartheta^3 \otimes \vartheta^3. \tag{4.1}$$

For the sake of clarity we repeat formula (4.1) giving the tensor indices explicitly: $g_{\alpha\beta} = \vartheta^0_\alpha \vartheta^0_\beta - \vartheta^1_\alpha \vartheta^1_\beta - \vartheta^2_\alpha \vartheta^2_\beta - \vartheta^3_\alpha \vartheta^3_\beta$.

Formula (4.1) means that the coframe is a field of orthonormal bases with orthonormality understood in the Lorentzian sense. Of course, at every point of the manifold M the choice of coframe is not unique: there are 6 real degrees of freedom in choosing the coframe and any pair of coframes is related by a Lorentz transformation.

As dynamical variables in our model we choose a coframe ϑ and a positive density ρ. These live in the original $(1+3)$-dimensional spacetime $\{M, g\}$ and are functions of local coordinates (x^0, x^1, x^2, x^3).

In order to incorporate into our model mass and electromagnetic field we perform a Kaluza–Klein extension: we add an extra coordinate x^4 and work on the resulting 5-manifold which we denote by $\mathbf{M}$. We suppose that

- the coordinate x^4 is fixed,
- we allow only changes of coordinates (x^0, x^1, x^2, x^3) which do not depend on x^4.

We will use **bold** type for extended quantities.

We extend our coframe as

$$\boldsymbol{\vartheta}^0_{\boldsymbol{\alpha}} = \begin{pmatrix} \vartheta^0_\alpha \\ 0 \end{pmatrix}, \qquad \boldsymbol{\vartheta}^3_{\boldsymbol{\alpha}} = \begin{pmatrix} \vartheta^3_\alpha \\ 0 \end{pmatrix}, \tag{4.2}$$

$$(\boldsymbol{\vartheta}^1 + i\boldsymbol{\vartheta}^2)_{\boldsymbol{\alpha}} = \begin{pmatrix} (\vartheta^1 + i\vartheta^2)_\alpha \\ 0 \end{pmatrix} e^{-2imx^4}, \tag{4.3}$$

$$\boldsymbol{\vartheta}^4_{\boldsymbol{\alpha}} = \begin{pmatrix} 0_\alpha \\ 1 \end{pmatrix} \tag{4.4}$$

where the bold tensor index $\boldsymbol{\alpha}$ runs through the values 0, 1, 2, 3, 4, whereas its non-bold counterpart α runs through the values 0, 1, 2, 3. In particular, the 0_α in formula (4.4) stands for a column of four zeros.

The coordinate x^4 parametrises a circle of radius $\frac{1}{2m}$. Condition (4.3) means that the extended coframe $\boldsymbol{\vartheta}$ experiences a full turn in the $(\vartheta^1, \vartheta^2)$-plane as we move along this circle, coming back to the starting point.

We extend our metric as

$$\mathbf{g}_{\boldsymbol{\alpha\beta}} := \begin{pmatrix} g_{\alpha\beta} - \frac{1}{m^2} A_\alpha A_\beta & \frac{1}{m} A_\alpha \\ \frac{1}{m} A_\beta & -1 \end{pmatrix}. \tag{4.5}$$

Formula (4.5) means that we view electromagnetism as a perturbation (shear) of the extended metric. Recall that in classical elasticity “shear” stands for “perturbation of the metric which does not change the volume”. It is easy to see that formula (4.5) implies $\det \mathbf{g} = -\det g$, so $\det \mathbf{g}$ does not depend on A and, hence, the electromagnetic field does not change the volume form.

Note that when $A \neq 0$ the extended coframe and the extended metric no longer agree:

$$\mathbf{g} \neq \boldsymbol{\vartheta}^0 \otimes \boldsymbol{\vartheta}^0 - \boldsymbol{\vartheta}^1 \otimes \boldsymbol{\vartheta}^1 - \boldsymbol{\vartheta}^2 \otimes \boldsymbol{\vartheta}^2 - \boldsymbol{\vartheta}^3 \otimes \boldsymbol{\vartheta}^3 - \boldsymbol{\vartheta}^4 \otimes \boldsymbol{\vartheta}^4 \tag{4.6}$$

(compare with (4.1)). The full physical implications of this discord are not discussed in the current paper. We need the extended metric only for raising tensor indices (see formula (4.9) below) and for this purpose the discord (4.6) is irrelevant.

We define the 3-form

$$\mathbf{T}^{\mathrm{ax}} := \frac{1}{3}(\boldsymbol{\vartheta}^0 \wedge d\boldsymbol{\vartheta}^0 - \boldsymbol{\vartheta}^1 \wedge d\boldsymbol{\vartheta}^1 - \boldsymbol{\vartheta}^2 \wedge d\boldsymbol{\vartheta}^2 - \boldsymbol{\vartheta}^3 \wedge d\boldsymbol{\vartheta}^3 - \underbrace{\boldsymbol{\vartheta}^4 \wedge d\boldsymbol{\vartheta}^4}_{=0}) \tag{4.7}$$

where d denotes the exterior derivative. This 3-form is called *axial torsion of the teleparallel connection.* An explanation of the geometric meaning of the latter phrase as well as a detailed exposition of the application of torsion in field theory and the history of the subject can be found in [3]. For our purposes the 3-form (4.7) is simply a measure of deformations generated by rotations of spacetime points.

We choose our Lagrangian density to be

$$L(\vartheta, \rho) := \|\mathbf{T}^{\mathrm{ax}}\|^2 \rho \tag{4.8}$$

where

$$\|\mathbf{T}^{\mathrm{ax}}\|^2 := \frac{1}{3!}\, \mathbf{T}^{\mathrm{ax}}_{\boldsymbol{\alpha\beta\gamma}}\, \mathbf{T}^{\mathrm{ax}}_{\boldsymbol{\kappa\lambda\mu}}\, \mathbf{g}^{\boldsymbol{\alpha\kappa}}\, \mathbf{g}^{\boldsymbol{\beta\lambda}}\, \mathbf{g}^{\boldsymbol{\gamma\mu}}\,. \tag{4.9}$$

Formula (4.3) implies

$$\boldsymbol{\vartheta}^1 \wedge d\boldsymbol{\vartheta}^1 + \boldsymbol{\vartheta}^2 \wedge d\boldsymbol{\vartheta}^2 = \vartheta^1 \wedge d\vartheta^1 + \vartheta^2 \wedge d\vartheta^2 - 4m\vartheta^1 \wedge \vartheta^2 \wedge \boldsymbol{\vartheta}^4 \tag{4.10}$$

so our Lagrangian density $L(\vartheta, \rho)$ does not depend on x^4 and can be viewed as a Lagrangian density in the original spacetime of dimension $1 + 3$. This means, essentially, that we have performed a separation of variables in a nonlinear setting.

Our action (variational functional) is $\int L(\vartheta, \rho)\, dx^0 dx^1 dx^2 dx^3$. Our field equations (Euler–Lagrange equations) are obtained by varying this action with respect to the coframe ϑ and density ρ. Varying with respect to the density ρ is easy: this gives the field equation $\|\mathbf{T}^{\mathrm{ax}}\|^2 = 0$ which is equivalent to $L(\vartheta, \rho) = 0$. Varying with respect to the coframe ϑ is more difficult because we have to maintain the metric constraint (4.1); recall that the metric is assumed to be prescribed (fixed).

We do not write down the field equations for the Lagrangian density $L(\vartheta, \rho)$ explicitly. We note only that they are highly nonlinear and do not appear to bear any resemblance to the linear Dirac equation (3.1).

5. Choosing a common language

In order to compare the two models described in Sections 3 and 4 we need to choose a common mathematical language. We choose the language of bispinors. Namely, we express the coframe and density via a bispinor field according to formulae

$$s = \xi^a \bar{\eta}_a, \tag{5.1}$$

$$\rho = |s| \sqrt{|\det |g|}\,, \tag{5.2}$$

$$(\vartheta^0 + \vartheta^3)_\alpha = |s|^{-1} \xi^a \sigma_{\alpha a \dot{b}} \bar{\xi}^{\dot{b}}, \tag{5.3}$$

$$(\vartheta^0 - \vartheta^3)_\alpha = |s|^{-1} \bar{\eta}^a \sigma_{\alpha a \dot{b}} \eta^{\dot{b}}, \tag{5.4}$$

$$(\vartheta^1 + i\vartheta^2)_\alpha = -|s|^{-1} \xi^a \sigma_{\alpha a \dot{b}} \eta^{\dot{b}} \tag{5.5}$$

where

$$\eta^{\dot{a}} = \epsilon^{\dot{a}\dot{b}} \eta_{\dot{b}}, \qquad \epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.6}$$

Note that throughout this paper we assume that the density ρ does not vanish.

Observe now that the right-hand sides of formulae (5.2)–(5.5) are invariant under the change of bispinor field $\xi^a \mapsto \xi^a e^{i\varphi}$, $\eta_{\dot{b}} \mapsto \eta_{\dot{b}} e^{-i\varphi}$ where $\varphi : M \to \mathbb{R}$ is an arbitrary scalar function. In other words, formulae (5.2)–(5.5) do not feel the argument of the complex scalar s. Hence, when translating our model into the language of bispinors it is natural to impose the constraint

$$\operatorname{Im} s = 0, \qquad s > 0. \tag{5.7}$$

This constraint reflects the fact that our model has one real dynamical degree of freedom less than the Dirac model (seven real degrees of freedom instead of eight).

6. Special case with no dependence on x^3

In addition to our usual assumptions (see beginning of Section 4) we suppose that

- the coordinate x^3 is fixed,
- we allow only changes of coordinates (x^0, x^1, x^2) which do not depend on x^3,
- the metric does not depend on x^3 and has block structure

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & 0 \\ g_{10} & g_{11} & g_{12} & 0 \\ g_{20} & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{6.1}$$

- the electromagnetic covector potential does not depend on x^3 and has $A_3 = 0$.

We work with coframes such that

$$\vartheta^3_\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6.2}$$

We use Pauli matrices which do not depend on x^3 and take

$$\sigma_{3a\dot{b}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.3}$$

We take

$$\eta_{\dot{b}} = \xi^a \sigma_{3a\dot{b}}\,. \tag{6.4}$$

Then the scalar defined by formula (5.1) takes the form $s = |\xi^1|^2 - |\xi^2|^2$. This scalar is automatically real and condition (5.7) becomes

$$|\xi^1|^2 - |\xi^2|^2 > 0. \tag{6.5}$$

It is easy to see that formulae (6.1), (6.3)–(6.5) imply (6.2).

Formula (6.4) means that our bispinor ξ^a, $\eta_{\dot{b}}$ is determined by the spinor ξ^a. Thus, the spinor ξ^a becomes the (only) dynamical variable. We assume that this spinor does not depend on x^3.

Observe that in the special case considered in this section both the Dirac model and our model have the same number of real dynamical degrees of freedom, namely, four. This is because under the assumption (6.2) the coframe ϑ and density ρ are equivalent to a spinor field ξ^a modulo sign ($-\xi^a$ gives the same ϑ and ρ).

Throughout this section summation is carried out either over indices 0, 1, 2 or over indices 0, 1, 2, 4. In the latter case we use **bold** type.

Put

$$L^\pm_{\mathrm{Dir}}(\xi) := \Big[\frac{i}{2}(\bar\xi^{\dot{b}} \sigma^\alpha{}_{a\dot{b}} \nabla_\alpha \xi^a - \xi^a \sigma^\alpha{}_{a\dot{b}} \nabla_\alpha \bar\xi^{\dot{b}}) + A_\alpha \xi^a \sigma^\alpha{}_{a\dot{b}} \bar\xi^{\dot{b}} \mp m \xi^a \sigma_{3a\dot{b}} \bar\xi^{\dot{b}}\Big] \sqrt{|\det g|}\,. \tag{6.6}$$

The Lagrangian densities $L^\pm_{\mathrm{Dir}}(\xi)$ are formally related to the original Lagrangian density (3.2) as follows: if we set $\eta_{\dot{b}} = \pm\xi^a \sigma_{3a\dot{b}}$ we get $L_{\mathrm{Dir}}(\xi, \eta) = 2L^\pm_{\mathrm{Dir}}(\xi)$. We say "formally related" because in this section we assume that formula $\eta_{\dot b} = \pm\xi^a\sigma_{3a\dot b}$ holds with upper sign, see (6.4). The $L^+_{\mathrm{Dir}}(\xi)$ and $L^-_{\mathrm{Dir}}(\xi)$ are, of course, the usual Dirac Lagrangian densities for an electron with spin up and spin down.

Lemma 6.1. *In the special case with no dependence on* x^3 *our Lagrangian density* (4.8) *factorises as*

$$L(\vartheta,\rho) = -\frac{32m}{9}\,\frac{L^+_{\mathrm{Dir}}(\xi)L^-_{\mathrm{Dir}}(\xi)}{L^+_{\mathrm{Dir}}(\xi) - L^-_{\mathrm{Dir}}(\xi)}\,. \tag{6.7}$$

Let us emphasize once again that throughout this paper we assume that the density ρ does not vanish. In the special case with no dependence on x^3 this assumption can be equivalently rewritten as

$$L^+_{\mathrm{Dir}}(\xi) \ne L^-_{\mathrm{Dir}}(\xi) \tag{6.8}$$

so the denominator in (6.7) is nonzero.

Proof. **Step 1.** Let us show that it is sufficient to prove formula (6.7) under the assumption $dA = 0$, i.e., under the assumption that the electromagnetic covector potential A is pure gauge. Recall that dA stands for the exterior derivative of A.

Suppose that we have already proved formula (6.7) under the assumption $dA = 0$ and are now looking at the case of general A. Let us fix an arbitrary point P on our 4-manifold M and prove formula (6.7) at this point. To do this, we perturb the electromagnetic covector potential A in such a way that

- A retains its value at the point P and
- A satisfies the condition $dA = 0$ in a neighbourhood of P.

This can be achieved by, say, choosing some local coordinates on M and setting the components of A to be constant in this coordinate system. Now, this perturbation of the covector potential A does not change the LHS or the RHS of (6.7) at the point P because neither of them depends on derivatives of A. Hence, the case of general A has been reduced to the case $dA = 0$.

Step 2. Let us show that it is sufficient to prove formula (6.7) under the assumption $A = 0$.

Suppose that we have already proved formula (6.7) under the assumption $A = 0$ and are now looking at the case $dA = 0$. Let us modify the definition of the extended coframe by replacing (4.4) with

$$\boldsymbol{\vartheta}^4_{\boldsymbol{\alpha}} = \begin{pmatrix} -\frac{1}{m}A_\alpha \\ 1 \end{pmatrix}. \tag{6.9}$$

In view of the condition $dA = 0$ this modification of the extended coframe does not change axial torsion (4.7) but the extended coframe (4.2), (4.3), (6.9) now agrees with the extended metric (4.5): we have

$$\mathbf{g} = \boldsymbol{\vartheta}^0 \otimes \boldsymbol{\vartheta}^0 - \boldsymbol{\vartheta}^1 \otimes \boldsymbol{\vartheta}^1 - \boldsymbol{\vartheta}^2 \otimes \boldsymbol{\vartheta}^2 - \boldsymbol{\vartheta}^3 \otimes \boldsymbol{\vartheta}^3 - \boldsymbol{\vartheta}^4 \otimes \boldsymbol{\vartheta}^4 \tag{6.10}$$

as opposed to (4.6). Let us now perform a change of coordinates

$$\tilde{x}^\alpha = x^\alpha, \quad \alpha = 0,1,2,3, \qquad \tilde{x}^4 = x^4 - \frac{1}{m}\int A \cdot dx. \tag{6.11}$$

Note that the integral $\int A \cdot dx$ is (locally) well defined because of the assumption $dA = 0$. The change of coordinates (6.11) is against the rules we stated in the beginning of Section 4 when describing our model (we changed the original Kaluza coordinate x^4 to a new coordinate $\tilde{x}^4$) but we are doing this only for the purpose of proving the lemma. In the new coordinate system $\tilde{x}$ the extended coframe (4.2), (4.3), (6.9) takes its original form (4.2)–(4.4), the extended metric takes the form $\mathbf{g}_{\boldsymbol{\alpha\beta}} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -1 \end{pmatrix}$ (compare with (4.5)) and the electromagnetic covector potential A is not affected (i.e., it has the same components in both coordinate systems). Observe now that in (4.3) we have retained the scalar factor e^{-2imx^4} written in terms of the original Kaluza coordinate x^4. Expressing x^4 in terms of $\tilde{x}^4$ in accordance with formula (6.11) we get

$$(\boldsymbol{\vartheta}^1 + i\boldsymbol{\vartheta}^2)_{\boldsymbol{\alpha}} = \begin{pmatrix} (\vartheta^1 + i\vartheta^2)_\alpha \\ 0 \end{pmatrix} e^{-2im\tilde{x}^4 - 2i\int A\cdot dx}. \tag{6.12}$$

Let us now introduce a new coframe $\hat{\vartheta}$ in (1+3)-dimensional spacetime $\{M, g\}$ related to the original coframe ϑ as

$$\hat{\vartheta}^0 = \vartheta^0, \qquad \hat{\vartheta}^3 = \vartheta^3, \qquad \hat{\vartheta}^1 + i\hat{\vartheta}^2 = (\vartheta^1 + i\vartheta^2)e^{-2i\int A\cdot dx}. \tag{6.13}$$

Then formulae (6.12), (6.13) imply

$$L(\hat{\vartheta}, \rho; 0) = L(\vartheta, \rho; A). \tag{6.14}$$

Here $L(\,\cdot\,,\,\cdot\,;\,\cdot\,)$ is our Lagrangian density $L(\,\cdot\,,\,\cdot\,)$ defined by formulae (4.2)–(4.5), (4.7)–(4.9) but with an extra entry after the semicolon for the electromagnetic covector potential. Formula (6.14) means that in our model the introduction of an electromagnetic covector potential A satisfying the condition $dA = 0$ is equivalent to a change of coframe (6.13).

Formulae (5.1)–(5.6), (6.4) imply that the change of coframe (6.13) leads to a change of spinor field $\hat{\xi}^a = \xi^a e^{-i\int A\cdot dx}$. Substituting the latter into (6.6) we get

$$L^\pm_{\mathrm{Dir}}(\hat{\xi}; 0) = L^\pm_{\mathrm{Dir}}(\xi; A). \tag{6.15}$$

Here $L^\pm_{\mathrm{Dir}}(\,\cdot\,;\,\cdot\,)$ is the Dirac Lagrangian density $L^\pm_{\mathrm{Dir}}(\,\cdot\,)$ defined by formula (6.6) but with an extra entry after the semicolon for the electromagnetic covector potential.

In the beginning of this part of the proof we assumed that we have already proved formula (6.7) under the assumption $A = 0$ so we have

$$L(\hat{\vartheta}, \rho; 0) = -\frac{32m}{9}\,\frac{L^+_{\mathrm{Dir}}(\hat{\xi}; 0)\,L^-_{\mathrm{Dir}}(\hat{\xi}; 0)}{L^+_{\mathrm{Dir}}(\hat{\xi}; 0) - L^-_{\mathrm{Dir}}(\hat{\xi}; 0)}\,. \tag{6.16}$$

It remains to note that formulae (6.14)–(6.16) imply (6.7). Hence, the case $dA = 0$ has been reduced to the case $A = 0$.

Step 3. In the remainder of the proof we assume that $A = 0$.

The proof of formula (6.7) is performed by direct substitution: it is just a matter of expressing the coframe and density via the spinor using formulae (5.1)–(5.6), (6.4) and substituting these expressions into the LHS of (6.7). However, even with $A = 0$ this is a massive calculation. In order to overcome these technical difficulties we perform below a trick which makes the calculations much easier. This trick is a known one and was, for example, extensively used by A. Dimakis and F. Müller-Hoissen [4, 5, 6].

Observe that when working with spinors we have the freedom in our choice of Pauli matrices: at every point of our $(1+3)$-dimensional spacetime $\{M, g\}$ we can apply a proper Lorentz transformation to a given set of Pauli matrices to get a new set of Pauli matrices, with spinor fields transforming accordingly. It is sufficient to prove formula (6.7) for one particular choice of Pauli matrices, hence it is natural to choose Pauli matrices in a way that makes calculations as simple as possible. We choose Pauli matrices

$$\sigma_{\alpha a\dot{b}} = \vartheta^j_\alpha\, s_{ja\dot{b}} = \vartheta^0_\alpha\, s_{0a\dot{b}} + \vartheta^1_\alpha\, s_{1a\dot{b}} + \vartheta^2_\alpha\, s_{2a\dot{b}} + \vartheta^3_\alpha\, s_{3a\dot{b}} \tag{6.17}$$

where

$$s_{ja\dot{b}} = \begin{pmatrix} s_{0a\dot{b}} \\ s_{1a\dot{b}} \\ s_{2a\dot{b}} \\ s_{3a\dot{b}} \end{pmatrix} := \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}. \tag{6.18}$$

Here ϑ is the coframe that appears in the LHS of formula (6.7). Let us stress that in the statement of the lemma Pauli matrices are not assumed to be related in any way to the coframe ϑ. We are just choosing the particular Pauli matrices (6.17), (6.18) to simplify calculations in our proof.

Examination of formulae (5.1)–(5.6), (6.4), (6.5), (6.17), (6.18) shows that with our special choice of Pauli matrices we have $\xi^2 = 0$ whereas ξ^1 is nonzero and real. We are about to write down the Dirac Lagrangian density (6.6) which is quadratic in ξ so the sign of ξ does not matter. So let $\xi^a = \begin{pmatrix} e^h \\ 0 \end{pmatrix}$ where $h : M \to \mathbb{R}$ is a scalar function. We get

$$\begin{aligned}
&\frac{i}{2}\bar{\xi}^{\dot{d}}\sigma^\alpha{}_{a\dot{d}}\nabla_\alpha\xi^a \\
&= \frac{i}{2}\bar{\xi}^{\dot{d}}(\sigma^\alpha{}_{a\dot{d}}\partial_\alpha h)\xi^a + \frac{i}{8}\bar{\xi}^{\dot{d}}\sigma^\alpha{}_{a\dot{d}}\sigma_\beta{}^{a\dot{c}}(\partial_\alpha\sigma^\beta{}_{b\dot{c}} + \Gamma^\beta{}_{\alpha\gamma}\sigma^\gamma{}_{b\dot{c}})\xi^b \\
&= \frac{ie^{2h}}{8}\sigma^\alpha{}_{a\dot{1}}\sigma_\beta{}^{a\dot{c}}(\partial_\alpha\sigma^\beta{}_{1\dot{c}} + \Gamma^\beta{}_{\alpha\gamma}\sigma^\gamma{}_{1\dot{c}}) + \cdots = \frac{ie^{2h}}{8}\sigma^\alpha{}_{a\dot{1}}\sigma_\beta{}^{a\dot{c}}\nabla_\alpha\sigma^\beta{}_{1\dot{c}} + \cdots
\end{aligned}$$

$$
\begin{aligned}
&= \frac{ie^{2h}}{8}\big[\sigma^\alpha{}_{1\dot 1}\sigma_\beta{}^{1\dot 1}\nabla_\alpha\sigma^\beta{}_{1\dot 1} + \sigma^\alpha{}_{1\dot 1}\sigma_\beta{}^{1\dot 2}\nabla_\alpha\sigma^\beta{}_{1\dot 2} \\
&\qquad + \sigma^\alpha{}_{2\dot 1}\sigma_\beta{}^{2\dot 1}\nabla_\alpha\sigma^\beta{}_{1\dot 1} + \sigma^\alpha{}_{2\dot 1}\sigma_\beta{}^{2\dot 2}\nabla_\alpha\sigma^\beta{}_{1\dot 2}\big] + \cdots \\
&= \frac{ie^{2h}}{8}\big[\vartheta^{0\alpha}\sigma_\beta{}^{1\dot 1}\nabla_\alpha\sigma^\beta{}_{1\dot 1} + \vartheta^{0\alpha}\sigma_\beta{}^{1\dot 2}\nabla_\alpha\sigma^\beta{}_{1\dot 2} \\
&\qquad + (\vartheta^1 - i\vartheta^2)^\alpha\sigma_\beta{}^{2\dot 1}\nabla_\alpha\sigma^\beta{}_{1\dot 1} + (\vartheta^1 - i\vartheta^2)^\alpha\sigma_\beta{}^{2\dot 2}\nabla_\alpha\sigma^\beta{}_{1\dot 2}\big] + \cdots \\
&= \frac{ie^{2h}}{8}\big[\vartheta^{0\alpha}\sigma_\beta{}^{1\dot 1}\nabla_\alpha\vartheta^{0\beta} + \vartheta^{0\alpha}\sigma_\beta{}^{1\dot 2}\nabla_\alpha(\vartheta^1 + i\vartheta^2)^\beta \\
&\qquad + (\vartheta^1 - i\vartheta^2)^\alpha\sigma_\beta{}^{2\dot 1}\nabla_\alpha\vartheta^{0\beta} + (\vartheta^1 - i\vartheta^2)^\alpha\sigma_\beta{}^{2\dot 2}\nabla_\alpha(\vartheta^1 + i\vartheta^2)^\beta\big] + \cdots \\
&= \frac{ie^{2h}}{8}\big[\vartheta^{0\alpha}\vartheta^0_\beta\nabla_\alpha\vartheta^{0\beta} - \vartheta^{0\alpha}(\vartheta^1 - i\vartheta^2)_\beta\nabla_\alpha(\vartheta^1 + i\vartheta^2)^\beta \\
&\qquad - (\vartheta^1 - i\vartheta^2)^\alpha(\vartheta^1 + i\vartheta^2)_\beta\nabla_\alpha\vartheta^{0\beta} + (\vartheta^1 - i\vartheta^2)^\alpha\vartheta^0_\beta\nabla_\alpha(\vartheta^1 + i\vartheta^2)^\beta\big] + \cdots \\
&= \frac{ie^{2h}}{8}\big[-i\vartheta^{0\alpha}\vartheta^1_\beta\nabla_\alpha\vartheta^{2\beta} + i\vartheta^{0\alpha}\vartheta^2_\beta\nabla_\alpha\vartheta^{1\beta} - i\vartheta^{1\alpha}\vartheta^2_\beta\nabla_\alpha\vartheta^{0\beta} \\
&\quad + i\vartheta^{2\alpha}\vartheta^1_\beta\nabla_\alpha\vartheta^{0\beta} + i\vartheta^{1\alpha}\vartheta^0_\beta\nabla_\alpha\vartheta^{2\beta} - i\vartheta^{2\alpha}\vartheta^0_\beta\nabla_\alpha\vartheta^{1\beta}\big] + \cdots \\
&= \frac{e^{2h}}{8}\big[\vartheta^{0\alpha}\vartheta^1_\beta\nabla_\alpha\vartheta^{2\beta} - \vartheta^{0\alpha}\vartheta^2_\beta\nabla_\alpha\vartheta^{1\beta} + \vartheta^{1\alpha}\vartheta^2_\beta\nabla_\alpha\vartheta^{0\beta} \\
&\qquad - \vartheta^{2\alpha}\vartheta^1_\beta\nabla_\alpha\vartheta^{0\beta} - \vartheta^{1\alpha}\vartheta^0_\beta\nabla_\alpha\vartheta^{2\beta} + \vartheta^{2\alpha}\vartheta^0_\beta\nabla_\alpha\vartheta^{1\beta}\big] + \cdots \\
&= \frac{s}{8}\big[(\vartheta^0 \wedge \vartheta^1)\cdot d\vartheta^2 + (\vartheta^1 \wedge \vartheta^2)\cdot d\vartheta^0 + (\vartheta^2 \wedge \vartheta^0)\cdot d\vartheta^1\big] + \cdots
\end{aligned}
$$

where the dots denote purely imaginary terms. Hence,

$$\frac{i}{2}(\bar\xi^{\dot b}\sigma^\alpha{}_{a\dot b}\nabla_\alpha\xi^a - \xi^a\sigma^\alpha{}_{a\dot b}\nabla_\alpha\bar\xi^{\dot b}) = \frac{s}{4}[(\vartheta^0 \wedge \vartheta^1)\cdot d\vartheta^2 + (\vartheta^1 \wedge \vartheta^2)\cdot d\vartheta^0 + (\vartheta^2 \wedge \vartheta^0)\cdot d\vartheta^1].$$

Formula (6.6) with $A = 0$ can now be rewritten as

$$L^\pm_{\mathrm{Dir}}(\xi) = \left[\frac{1}{4}\big[(\vartheta^0 \wedge \vartheta^1)\cdot d\vartheta^2 + (\vartheta^1 \wedge \vartheta^2)\cdot d\vartheta^0 + (\vartheta^2 \wedge \vartheta^0)\cdot d\vartheta^1\big] \mp m\right]\rho\,. \quad (6.19)$$

Put

$$T^{\mathrm{ax}} := \frac{1}{3}(\vartheta^0 \wedge d\vartheta^0 - \vartheta^1 \wedge d\vartheta^1 - \vartheta^2 \wedge d\vartheta^2 - \underbrace{\vartheta^3 \wedge d\vartheta^3}_{=0}) \quad (6.20)$$

(compare with (4.7)). The last term in (6.20) vanishes in view of (6.2). The coordinate x^3 is redundant so T^{ax} can be viewed as a 3-form in $(1+2)$-dimensional Lorentzian space with local coordinates (x^0, x^1, x^2). Hence, we can define the scalar

$$*T^{\mathrm{ax}} := \frac{1}{3!}\sqrt{|\det g|}\,(T^{\mathrm{ax}})^{\alpha\beta\gamma}\varepsilon_{\alpha\beta\gamma} \quad (6.21)$$

which is the Hodge dual of T^{ax}. But $\sqrt{|\det g|}\,\varepsilon_{\alpha\beta\gamma}=(\vartheta^0\wedge\vartheta^1\wedge\vartheta^2)_{\alpha\beta\gamma}$ so formula (6.21) can be rewritten as

$$*T^{\mathrm{ax}}=T^{\mathrm{ax}}\cdot(\vartheta^0\wedge\vartheta^1\wedge\vartheta^2)=\frac{1}{3}(\vartheta^0\wedge d\vartheta^0-\vartheta^1\wedge d\vartheta^1-\vartheta^2\wedge d\vartheta^2)\cdot(\vartheta^0\wedge\vartheta^1\wedge\vartheta^2)$$
$$=\frac{1}{3}\big[(\vartheta^0\wedge\vartheta^1)\cdot d\vartheta^2+(\vartheta^1\wedge\vartheta^2)\cdot d\vartheta^0+(\vartheta^2\wedge\vartheta^0)\cdot d\vartheta^1\big].$$

Substituting the latter into (6.19) we arrive at the compact formula

$$L^{\pm}_{\mathrm{Dir}}(\xi)=\left[\frac{3}{4}*T^{\mathrm{ax}}\mp m\right]\rho\,. \tag{6.22}$$

Substituting (6.22) into the RHS of (6.7) we get

$$-\frac{32m}{9}\,\frac{L^{+}_{\mathrm{Dir}}(\xi)L^{-}_{\mathrm{Dir}}(\xi)}{L^{+}_{\mathrm{Dir}}(\xi)-L^{-}_{\mathrm{Dir}}(\xi)}=\left[(*T^{\mathrm{ax}})^2-\frac{16}{9}m^2\right]\rho\,.$$

As our Lagrangian $L(\vartheta,\rho)$ is defined by formula (4.8), the proof of the lemma has been reduced to proving

$$\|\mathbf{T}^{\mathrm{ax}}\|^2=(*T^{\mathrm{ax}})^2-\frac{16}{9}m^2 \tag{6.23}$$

with $A=0$ (recall that A initially appeared in the extended metric (4.5)).

In view of (4.2), (6.2) formula (4.7) becomes

$$\mathbf{T}^{\mathrm{ax}}=\frac{1}{3}(\boldsymbol{\vartheta}^0\wedge d\boldsymbol{\vartheta}^0-\boldsymbol{\vartheta}^1\wedge d\boldsymbol{\vartheta}^1-\boldsymbol{\vartheta}^2\wedge d\boldsymbol{\vartheta}^2). \tag{6.24}$$

The difference between formulae (6.20) and (6.24) is that the RHS of (6.20) is expressed via the coframe ϑ in the original spacetime whereas the RHS of (6.24) is expressed via the coframe $\boldsymbol{\vartheta}$ in the extended spacetime, see (4.2)–(4.4). In view of (4.10), (6.20) formula (6.24) can be rewritten as

$$\mathbf{T}^{\mathrm{ax}}=T^{\mathrm{ax}}+\frac{4m}{3}\vartheta^1\wedge\vartheta^2\wedge\boldsymbol{\vartheta}^4. \tag{6.25}$$

The coordinate x^3 is redundant so $\mathbf{T}^{\mathrm{ax}}$ can be viewed as a 3-form in $(1+3)$-dimensional Lorentzian space with local coordinates (x^0,x^1,x^2,x^4). Hence, we can define the covector

$$(*\mathbf{T}^{\mathrm{ax}})_{\boldsymbol{\delta}}:=\frac{1}{3!}\sqrt{|\det g|}\,(\mathbf{T}^{\mathrm{ax}})^{\boldsymbol{\alpha\beta\gamma}}\varepsilon_{\boldsymbol{\alpha\beta\gamma\delta}}\,,\qquad \boldsymbol{\delta}=0,1,2,4, \tag{6.26}$$

which is the Hodge dual of $\mathbf{T}^{\mathrm{ax}}$. It is easy to see that we have

$$\|\mathbf{T}^{\mathrm{ax}}\|^2=-\|*\mathbf{T}^{\mathrm{ax}}\|^2. \tag{6.27}$$

Note that in the LHS of (6.27) we square a 3-form in (1+4)-dimensional Lorentzian space whereas in the RHS of (6.27) we square a 1-form in (1 + 3)-dimensional Lorentzian space, so we took great care in getting the sign right. Substituting (6.25) into (6.26) we get

$$(*\mathbf{T}^{\mathrm{ax}})_{\boldsymbol{\delta}}=\begin{pmatrix}\frac{4m}{3}\vartheta^0_\delta\\ *T^{\mathrm{ax}}\end{pmatrix} \tag{6.28}$$

where $*T^{\mathrm{ax}}$ is the scalar defined by formula (6.21). It remains to observe that formulae (6.27), (6.28) imply (6.23). □

The following theorem is the main result of our paper.

Theorem 6.2. *In the special case with no dependence on x^3 a coframe ϑ and a density ρ are a solution of the field equations for the Lagrangian density $L(\vartheta,\rho)$ if and only if the corresponding spinor field is a solution of the field equation for the Lagrangian density $L^+_{\mathrm{Dir}}(\xi)$ or the field equation for the Lagrangian density $L^-_{\mathrm{Dir}}(\xi)$.*

Proof. Denote by $L(\xi)$ the Lagrangian density (4.8) but with ϑ and ρ expressed via ξ. Accordingly, we rewrite the factorisation formula (6.7) as

$$L(\xi) = -\frac{32m}{9}\,\frac{L^+_{\mathrm{Dir}}(\xi)L^-_{\mathrm{Dir}}(\xi)}{L^+_{\mathrm{Dir}}(\xi) - L^-_{\mathrm{Dir}}(\xi)}\,. \tag{6.29}$$

Observe also that the Dirac Lagrangian densities $L^\pm_{\mathrm{Dir}}$ defined by formula (6.6) possess the property of scaling covariance:

$$L^\pm_{\mathrm{Dir}}(e^h\xi) = e^{2h}L^\pm_{\mathrm{Dir}}(\xi) \tag{6.30}$$

where $h: M \to \mathbb{R}$ is an arbitrary scalar function.

We claim that the statement of the theorem follows from (6.29) and (6.30). The proof presented below is an abstract one and does not depend on the physical nature of the dynamical variable ξ, the only requirement being that it is an element of a vector space so that scaling makes sense.

Note that formulae (6.29) and (6.30) imply that the Lagrangian density L possesses the property of scaling covariance, so all three of our Lagrangian densities, L, L^+_{Dir} and L^-_{Dir}, have this property. Note also that if ξ is a solution of the field equation for some Lagrangian density $\mathcal{L}$ possessing the property of scaling covariance then $\mathcal{L}(\xi) = 0$. Indeed, let us perform a scaling variation of our dynamical variable

$$\xi \mapsto \xi + h\xi \tag{6.31}$$

where $h: M \to \mathbb{R}$ is an arbitrary "small" scalar function with compact support. Then $0 = \delta\int\mathcal{L}(\xi) = 2\int h\mathcal{L}(\xi)$ which holds for arbitrary h only if $\mathcal{L}(\xi) = 0$.

In the remainder of the proof the variations of ξ are arbitrary and not necessarily of the scaling type (6.31).

Suppose that ξ is a solution of the field equation for the Lagrangian density L^+_{Dir}. [The case when ξ is a solution of the field equation for the Lagrangian density L^-_{Dir} is handled similarly.] Then $L^+_{\mathrm{Dir}}(\xi) = 0$ and, in view of (6.8), $L^-_{\mathrm{Dir}}(\xi) \neq 0$. Varying ξ, we get

$$\begin{aligned}\delta\int L(\xi) &= -\frac{32m}{9}\Big(\int \frac{L^-_{\mathrm{Dir}}(\xi)}{L^+_{\mathrm{Dir}}(\xi) - L^-_{\mathrm{Dir}}(\xi)}\delta L^+_{\mathrm{Dir}}(\xi) + \int L^+_{\mathrm{Dir}}(\xi)\delta\frac{L^-_{\mathrm{Dir}}(\xi)}{L^+_{\mathrm{Dir}}(\xi)-L^-_{\mathrm{Dir}}(\xi)}\Big)\\ &= \frac{32m}{9}\int \delta L^+_{\mathrm{Dir}}(\xi) = \frac{32m}{9}\,\delta\int L^+_{\mathrm{Dir}}(\xi)\end{aligned}$$

so

$$\delta \int L(\xi) = \frac{32m}{9}\, \delta \int L^+_{\mathrm{Dir}}(\xi)\,. \tag{6.32}$$

We assumed that ξ is a solution of the field equation for the Lagrangian density L^+_{Dir} so $\delta \int L^+_{\mathrm{Dir}}(\xi) = 0$ and formula (6.32) implies that $\delta \int L(\xi) = 0$. As the latter is true for an arbitrary variation of ξ this means that ξ is a solution of the field equation for the Lagrangian density L.

Suppose that ξ is a solution of the field equation for the Lagrangian density L. Then $L(\xi) = 0$ and formula(6.29) implies that either $L^+_{\mathrm{Dir}}(\xi) = 0$ or $L^-_{\mathrm{Dir}}(\xi) = 0$; note that in view of (6.8) we cannot have simultaneously $L^+_{\mathrm{Dir}}(\xi) = 0$ and $L^-_{\mathrm{Dir}}(\xi) = 0$. Assume for definiteness that $L^+_{\mathrm{Dir}}(\xi) = 0$. [The case when $L^-_{\mathrm{Dir}}(\xi) = 0$ is handled similarly.] Varying ξ and repeating the argument from the previous paragraph we arrive at (6.32). We assumed that ξ is a solution of the field equation for the Lagrangian density L so $\delta \int L(\xi) = 0$ and formula (6.32) implies that $\delta \int L^+_{\mathrm{Dir}}(\xi) = 0$. As the latter is true for an arbitrary variation of ξ this means that ξ is a solution of the field equation for the Lagrangian density L^+_{Dir}. □

The proof of Theorem 6.2 presented above may appear to be non-rigorous but it can be easily recast in terms of explicitly written field equations.

7. Discussion

The mathematical model formulated in Section 4 is based on the idea that every point of spacetime can rotate and that rotations of different points are totally independent. The idea of studying such continua belongs to the Cosserat brothers [7]. Recall that in classical elasticity the deformation of a continuum is described by a (co)vector function u, the field of displacements, which is the dynamical variable (unknown quantity) in the system of equations. Displacements, of course, generate rotations: the infinitesimal rotation caused by a displacement field u is du, the exterior derivative of u. The Cosserat brothers' idea was to make rotations totally independent of displacements, so that the coframe (field of orthonormal bases attached to points of the continuum) becomes an additional dynamical variable.

Our model is a special case of Cosserat elasticity in that we model spacetime as a continuum which cannot experience displacements, only rotations. The idea of studying such continua is also not new: it lies at the heart of the theory of *teleparallelism* (= absolute parallelism), a subject promoted in the end of the 1920s by A. Einstein and É. Cartan [8, 9, 10]. It is interesting that Einstein pursued this activity precisely with the aim of modelling the electron, but, unfortunately, without success.

The differences between our mathematical model formulated in Section 4 and mathematical models commonly used in teleparallelism are as follows.

- We assume the metric to be prescribed (fixed) whereas in teleparallelism it is traditional to view the metric as a dynamical variable. In other words, in works on teleparallelism it is customary to view (4.1) not as a constraint but as a definition of the metric and, consequently, to vary the coframe without any constraints at all. This is not surprising as most, if not all, authors who contributed to teleparallelism came to the subject from General Relativity.

- We choose a very particular Lagrangian density (4.8) containing only one irreducible piece of torsion (axial) whereas in teleparallelism it is traditional to choose a more general Lagrangian containing all three pieces (tensor, trace and axial), see formula (26) in [3].

We now explain the motivation behind our choice of the Lagrangian density (4.8). Suppose for simplicity that we don't have electromagnetism, i.e., that $A = 0$, in which case the extended coframe and extended metric agree (6.10). Let us perform a conformal rescaling of the extended coframe: $\boldsymbol{\vartheta}^j \mapsto e^h \boldsymbol{\vartheta}^j$, $j = 0, 1, 2, 3, 4$, where $h : M \to \mathbb{R}$ is an arbitrary scalar function. Then the metric and axial torsion scale as $\mathbf{g} \mapsto e^{2h}\mathbf{g}$ and

$$\mathbf{T}^{\mathrm{ax}} \mapsto e^{2h}\mathbf{T}^{\mathrm{ax}} \tag{7.1}$$

respectively. Here the remarkable fact is that the derivatives of h do not appear in formula (7.1) which means that axial torsion is the irreducible piece of torsion which is conformally covariant. It remains to note that if we scale the density ρ as $\rho \mapsto e^{2h}\rho$ then the Lagrangian density (4.8) will not change.

Thus, the guiding principle in our choice of the Lagrangian density (4.8) is conformal invariance. This does not, however, mean that our mathematical model formulated in Section 4 is conformally invariant: formula (4.4) does not allow for conformal rescalings. The Kaluza–Klein extension is a procedure which breaks conformal invariance, as one would expect when introducing mass.

The main result of our paper is Theorem 6.2 which establishes that in the special case with no dependence on x^3 our mathematical model is equivalent to the Dirac equation. This special case is known in literature as the Dirac equation in dimension $1 + 2$ and is in itself the subject of extensive research.

This leaves us with the question what can be said about the general case, when there is dependence on all spacetime coordinates (x^0, x^1, x^2, x^3). In the general case our model is clearly not equivalent to the Dirac equation because it lacks one real dynamical degree of freedom, see last paragraph in Section 5. Our plan for the future is to examine *how much* our model differs from the Dirac model in the general case. We plan to compare the two models by calculating energy spectra of the electron in a given stationary electromagnetic field, starting with the case of the Coulomb potential (hydrogen atom).

The spectral-theoretic analysis of our model will, however, pose a monumental analytic challenge. There are several fundamental issues that have to be dealt with.

- Our model does not appear to fit into the standard scheme of strongly hyperbolic systems of partial differential equations.
- The eigenvalue (= bound state) problem for our model is nonlinear.
- Our construction relies on the density ρ being strictly positive. This assumption may fail when one seeks bound states other than the ground state.

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Hölder and Lipschitz Estimates for Viscosity Solutions of Some Degenerate Elliptic PDE's

Italo Capuzzo Dolcetta

To Prof. V. Maz'ya, with great admiration for his mathematical achievements

Abstract. We report here on some recent results, obtained in collaboration with F. Leoni and A. Porretta [7] concerning Hölder and Lipschitz regularity and the solvability of the Dirichlet problem for degenerate quasilinear elliptic equations of the form

$$-\mathrm{Tr}\left(A(x)D^2u\right) + |Du|^p + \lambda u = f(x)\,,\; x \in \Omega\,.$$

The research presented here is partly motivated by a paper by J.M. Lasry and P.L. Lions [12]. Our results can be regarded as extensions to the degenerate elliptic case of some of those contained in that paper.

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1. Introduction

This note is about some recent results on gradient and Hölder estimates for viscosity solutions of some second-order *degenerate elliptic* partial differential equations. The model equation discussed here is

$$-\mathrm{Tr}\left(A(x)D^2u\right) + |Du|^p + \lambda u = f(x)\,,\; x \in \Omega\,. \tag{1.1}$$

In the above, $\Omega \subset \mathbb{R}^n$ is a bounded open set, $A(x)$ is a *non-negative definite* $n \times n$ symmetric matrix, Tr denotes its trace, D^2 and D are, respectively, the Hessian and the gradient of the unknown function $u : \Omega \to \mathbb{R}$, $p > 1$ and $\lambda \geq 0$ are given real numbers. We shall assume that $x \to A(x)$ and $x \to f(x)$ are continuous functions.

Equations of the form (1.1) arise for example as Dynamic Programming optimality conditions for some stochastic optimal control problems. Indeed, let us consider the *degenerate diffusion process* modelized by the Ito's stochastic differential equation

$$dX_t = a(X_t)\,dt + \sqrt{2A(X_t)}\,dW_t\,, \quad X_0 = x \in \Omega\,,$$

where W_t is a standard n-dimensional Brownian motion and $a(X_t)$, with $|a| \leq 1$, is a *feedback control* acting on the trajectory X_t. Define next the *cost functional*

$$J(x,a) = \mathbb{E}_x \left\{ \int_0^{\tau_x} [f(X_t) + c_q|a(X_t)|^q]\,\mathrm{e}^{-\lambda t}\,dt + g(X_{\tau_x})e^{-\lambda\tau_x} \right\} \tag{1.2}$$

where $\mathbb{E}_x$ is the conditional expectation with respect to the initial state $x = X_0$.

In (1.2), f, g are given continuous functions, $q > 1$, $c_q = \frac{1}{q}\left(1 - \frac{1}{q}\right)^{q-1}$ and $\lambda \geq 0$ represents a *discount rate.* Two cases of interest for the applications are when

$$\tau_x = \inf\left\{t \geq 0 : X_t \in \mathbb{R}^n \setminus \overline{\Omega}\right\} \quad \text{(the } \textit{exit time problem}\text{)}$$

$$\text{or} \qquad \tau_x = +\infty \quad \text{(the } \textit{state-constrained} \text{ problem).}$$

The optimal control problem associated to these data is to minimize, for each given initial position $X_0 = x$, the functional $J(x,a)$ with respect to feedback controls a belonging to a specified set $\mathcal{A}_x$ of *admissible controls.*

Dynamic Programming arguments show that even if the value function

$$u(x) = \inf_{a \in \mathcal{A}_x} J(x,a)$$

is just a continuous function on Ω (which is true under mild assumptions on the data of the problem), nonetheless it is a *viscosity solution* of equation (1.1) with $p = \frac{q}{q-1}$. Let us recall for the convenience of the reader that this means that

$$-\mathrm{Tr}\left(A(x_1)D^2\Phi(x_1)\right) + |D\Phi(x_1)|^p + \lambda\,\Phi(x_1) \leq f(x_1) \tag{1.3}$$

at any point $x_1 \in \Omega$ and for all C^2 function Φ touching from above the graph of u at x_1 (the *subsolution* condition) and, symmetrically,

$$-\mathrm{Tr}\left(A(x_0)D^2\Psi(x_0)\right) + |D\Psi(x_0)|^p + \lambda\,\Psi(x_0) \geq f(x_0) \tag{1.4}$$

at any point $x_0 \in \Omega$ and for all C^2 function Ψ touching from below the graph of u at x_0 (the *supersolution* condition).

Definitions (1.3), (1.4) make sense, respectively, for upper (lower) semicontinuous functions u. Moreover, u satisfies a boundary condition either of Dirichlet type

$$u = g \text{ on } \partial\Omega$$

for the exit-time problem, or of the quite unusual form

$$-\mathrm{Tr}(A(x)D^2u) + |Du|^p + \lambda\,u \geq f \text{ on } \partial\Omega$$

for the state-constrained problem. It can be shown also that the feedback control given by

$$a(x) = -p\,|Du(x)|^{p-2}Du(x)$$

is an *optimal* one, at least at those points where u is differentiable.

We refer to [10] and [1] for Dynamic Programming methods in optimal control problems and to [8] as a general reference text on viscosity solutions.

A deep and fairly exhaustive analysis of the state-constrained problem for equation (1.1) in the model non-degenerate case

$$-\Delta u + |Du|^p + \lambda u = f(x)\,,\ x \in \Omega \tag{1.5}$$

has been carried on by J.M. Lasry and P.L. Lions in [12]. In that situation, the authors prove that the value function u is is locally Lipschitz continuous and it is the maximal solution of (1.1). Concerning the boundary behavior, if $1 < p \leq 2$ then u *blows-up* on $\partial\Omega$, while if $p > 2$, then u is *bounded and Hölder continuous* up to $\partial\Omega$. This striking difference between the cases $p \leq 2$ and $p > 2$ reflects a similar feature occurring in the study of the exit-time problem, see [4]. Namely if $1 < p \leq 2$, the Dirichlet problem can be solved in the classical sense for any boundary datum g, while if $p > 2$ this is no longer true and there can be loss of boundary condition. In that case, the best one can expect, in general, is that the Dirichlet condition $u = g$ is satisfied only in a generalized sense, see [4].

Under some conditions, involving the possible degeneracy of $A(x)$ at points x of the boundary and a compatibility between the source term f and the boundary datum g, it is possible to prove the existence of solutions which satisfy the boundary condition at all points of $\partial\Omega$, see Theorem 4.1.

The setting of this note concerns the case when A is *non-negative definite* and the gradient term is *superquadratic*. The main results, obtained in collaboration with F. Leoni and A. Porretta [6, 7], are Theorem 2.1 about Hölder estimates for *bounded viscosity subsolutions* and Theorem 3.1 about gradient estimates for *bounded viscosity solutions* of equation (1.1). Finally, Theorem 4.1 is a result about the existence of solutions for the Dirichlet problem.

2. Local and global Hölder estimates for subsolutions

The main result in this section establishes the *global* Hölder continuity of bounded upper semicontinuous (USC, in short) *viscosity subsolutions* of equation

$$-\mathrm{Tr}\left(A(x)D^2u\right) + |Du|^p + \lambda u \leq f(x)\,,\ x \in \Omega.$$

The bound will depend in particular on the distance function from $\partial\Omega$, denoted by $d_{\partial\Omega}$.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be an open bounded set with Lipschitz boundary satisfying the uniform interior sphere condition. Let $u \in USC(\Omega)$ be any bounded viscosity solution of*

$$-\mathrm{Tr}\left(A(x)D^2u\right) + |Du|^p + \lambda u \leq f(x)\,,\ x \in \Omega\,, \tag{2.1}$$

with $\lambda \geq 0$ and $p > 2$. Assume further that $A(\cdot)$ is bounded and continuous, $f(\cdot)$ is continuous and that $d_{\partial\Omega}^{\frac{p}{p-1}}\, f^+$ is bounded.

Then, u is Hölder continuous in $\overline{\Omega}$ *and*

$$|u(x) - u(y)| \leq M\,|x-y|^{\frac{p-2}{p-1}} \qquad \forall\, x,\, y \in \overline{\Omega}$$

for some positive constant M *depending only on* $\partial\Omega$, p, $\|A\|_{L^\infty(\Omega)}$ *and*

$$\|d_{\partial\Omega}^{p/p-1}(f^+ + \lambda u^-)\|_{L^\infty(\Omega)}\,.$$

Before sketching the main steps of the proof, let us make some remarks on the above result.

Remark 2.2. As far as the Hölder exponent is concerned, the value $\frac{p-2}{p-1}$ is the best one can expect in the assumptions of the above Theorem. Indeed, a direct computation shows that the function $u(x) = |x|^{\frac{p-2}{p-1}}$ satisfies

$$-\Delta u + |Du|^p + \lambda u \leq \lambda \qquad \text{in} \;\; B_1(0) \subset \mathbb{R}^n$$

as soon as $n \geq 2$.

Remark 2.3. In the model case non-degenerate case (1.5), the Hölder continuity for bounded *solutions* was previously established in [12]. Note that our result above holds in fact for the wider class of *subsolutions*, a quite unusual feature for second-order pde's. Let us point out in this respect that in the special case $A \equiv 0$, the proof of Theorem 2.1 shows that the Hölder exponent is 1, in accordance with what is known about the Lipschitz character, see [2],[1] of subsolutions of the coercive Hamilton-Jacobi equation

$$|Du|^p + \lambda u \leq f(x)\,,\; x\, \in \Omega\,.$$

Remark 2.4. The a priori estimate on the Hölder seminorm of u depends only on $\|\lambda\, u^-\|_\infty$ and not on $\|u\|_\infty$. This fact is important when performing the *ergodic limit* $\lambda \to 0^+$ in equation (2.1) which is a basic tool in the analysis of homogenization problems, see [16] for recent results in this direction.

Remark 2.5. The linearity of the principal part in equation (2.1) does not play any role. The same result holds indeed if $u \in USC(\Omega)$ is a bounded viscosity solution of

$$F(x, Du, D^2u) + |Du|^p + \lambda u \,= f(x)\,, \qquad x \in \Omega\,,$$

with λ, p, f as in the statement of Theorem 2.1 and $F(x,\xi,M)$ is a continuous, *degenerate elliptic* function, i.e.,

$$F(x,\xi,M_1) - F(x,\xi,M_2) \leq 0 \qquad \forall x \in \Omega,\, \xi \in \mathbb{R}^n,\, M_1, M_2 \in \mathcal{S}_n^+ \text{ with } M_1 \geq M_2$$

satisfying, for some positive constants Λ, $\gamma > 0$

$$F(x,\xi,M) \geq -\Lambda\,\|M\| \qquad \forall x \in \Omega,\, \xi \in \mathbb{R}^n \,:\, |\xi| > \gamma,\, M \in \mathcal{S}_n^+\,.$$

Here we denoted by $\mathcal{S}_n^+$ the cone of non-negative definite symmetric $n \times n$ matrices.

The proof of Theorem 2.1 is technically quite involved (note that we are dealing with just upper semicontinuous subsolutions) and we only indicate here the main ideas while referring to [7] for full details.

The first step is to establish an *interior Hölder estimate*:

Proposition 2.6. *Under the same assumptions of Theorem* 2.1*, let B any open ball in Ω and let $d_{\partial B}$ denote the distance function from the boundary of B.*

If $u \in USC(B)$ is a bounded viscosity subsolution of (2.1) *in B, then*

$$u(x) - u(y) \leq K\left(\frac{|x-y|}{d_{\partial B}^{\frac{1}{p-1}}(x)} + |x-y|^{\frac{p-2}{p-1}}\right) \qquad \forall\, x,\, y \in B \tag{2.2}$$

where K is a positive constant depending only on p, $\|d_{\partial B}^{\frac{p}{p-1}}(f^+ + \lambda u^-)\|_{L^\infty(B)}$ and $\|A\|_{L^\infty(B)}$.

The proof makes use of the auxiliary function

$$w_{C,y,L}(x) = u(x) - u(y) - C\left(L\,|x-y|^{\frac{p-2}{p-1}} + \frac{|x-y|}{d_{\partial B}^{\frac{1}{p-1}}(x)}\right)$$

for fixed $y \in B$ and parameters $C > 0, L$ to be appropriately chosen.

The key point in the proof is to show that

$$M_{C,y,L} = \sup_{x\in B} w_{C,y,L}(x) \leq 0$$

for large enough $C, L > 0$. Assuming by contradiction that

$$M_{C,y,L} > 0 \tag{2.3}$$

it is not hard to show that the sup in the above relation is attained at some point $\hat{x} \in B$, $\hat{x} \neq y$. The function

$$\Phi(x) = u(y) + C\left(L\,|x-y|^{\frac{p-2}{p-1}} + \frac{|x-y|}{d_{\partial B}^{\frac{1}{p-1}}(x)}\right)$$

is an admissible test function in a neighborhood of $\hat{x}$, and the maximum of $w_{C,y,L}(x)$ is achieved inside B. From the definition of viscosity subsolution it follows that

$$-\mathrm{Tr}\left(A(\hat{x})D^2\Phi(\hat{x})\right) + |D\Phi(\hat{x})|^p \leq f(\hat{x}) - \lambda\, u(\hat{x}) \leq f^+(\hat{x}) + \lambda\, u^-(\hat{x})\,.$$

If (2.3) holds, a quite technical argument involving the computation of $D\Phi\,, D^2\Phi$ and exploiting in particular the facts that A is non–negative definite, $d_{\partial B}$ is non-negative and *concave*, $0 < \frac{1}{p-1} < 1$, leads to the inequality

$$\mathrm{Tr}\left(A(x)D^2\Phi(x)\right) \leq c\,C\|A\|_{L^\infty(B)}\,\Gamma(x,y)$$

for some constant $c > 0$, where

$$\Gamma(x,y) = \frac{d_{\partial B}^{\alpha-1}(x)}{|x-y|} + d_{\partial B}^{\alpha-2}(x) + |x-y|\,d_{\partial B}^{\alpha-3}(x) + L|x-y|^{\alpha-2}$$

with $\alpha = \frac{p-2}{p-1}$. Using again (2.1) we deduce, after some computations, that for

$$C \gg \|d_{\partial B}^{p(1-\alpha)}\left(f^{+} + \lambda u^{-}\right)\|_{L^{\infty}(B)}^{1/p}$$

a contradiction arises, showing that (2.3) cannot hold.

At this point it is not difficult to conclude the proof of the statement.

Observe that, by reversing the role of x and y in inequality (2.2), any subsolution u satisfies in fact

$$|u(x) - u(y)| \leq K\left[\frac{|x-y|}{(d_{\partial B}(x) \wedge d_{\partial B}(y))^{\frac{1}{p-1}}} + |x-y|^{\frac{p-2}{p-1}}\right].$$

This implies of course that u is actually continuous in B. The Hölderianity up to $\partial\Omega$ can be obtained under smoothness assumptions on the boundary of Ω.

At this purpose, we make use of the next technical lemma:

Lemma 2.7. *If $u : B \to \mathbb{R}$ is a continuous function such that*

$$|u(x) - u(y)| \leq K\left[\frac{|x-y|}{(d_{\partial B}(x) \wedge d_{\partial B}(y))^{1-\alpha}} + |x-y|^{\alpha}\right]$$

for some $K > 0$ and $0 < \alpha < 1$, then u can be extended up to $\overline{B}$ as a function satisfying

$$|u(x) - u(y)| \leq K'\,|x-y|^{\alpha} \quad \textit{for all} \quad x, y \in \overline{B}$$

where $K' \geq K$ is a constant depending on α, K.

We refer to [7] for the proof.

3. Lipschitz estimates

In the paper [12] the authors consider C^2 solutions of the model equation

$$-\Delta u + |Du|^p + \lambda u = f(x)\,,\; x \in \Omega\,. \tag{3.1}$$

Since the nonlinearity in the gradient slot is *convex*, their analysis based on the Bernstein method yields the local Lipschitz estimate

$$|Du(x)| \leq C, \quad x \in B \subset\subset \Omega\,, \tag{3.2}$$

for some $C > 0$ depending on the data f and λ. Once it is known that any solution is locally Lipschitz, then the following precise estimate

$$|Du(x)| \leq C \Big/ d_{\partial\Omega}^{\frac{1}{p-1}}(x)$$

can be derived by using a scaling argument relying on the homogeneity of the equation. More precisely, assume that u satisfies (3.1) and, for fixed $x_0 \in \Omega$, let $r = d_{\partial\Omega}(x_0)$. The scaled function

$$v(x) = \frac{1}{r^{\alpha}} u(x_0 + rx) \qquad x \in B_1(0)\,,$$

with $\alpha = \frac{p-2}{p-1}$ satisfies

$$-\Delta v + |Dv|^p + \lambda\, r^2 v = r^{2-\alpha} f(x_0 + rx) \qquad x \in B_1(0)\,.$$

One can use at this point the previously obtained Lipschitz bound (3.2) for function v. Hence, $|Dv(0)| \leq C$, that is

$$|Du(x_0)| \leq \frac{C}{d_{\partial\Omega}^{\frac{1}{p-1}}(x_0)} \, .$$

The same Lipschitz estimate can be proved for merely continuous functions satisfying equation (3.1) in the viscosity sense. Indeed we have the following result from [7]:

Theorem 3.1. *Let $u \in C(\Omega)$ be a bounded viscosity solution of*

$$-\mathrm{Tr}\left(A(x)D^2u\right) + |Du|^p + \lambda u = f(x)\,, \qquad x \in \Omega\,,$$

with $p > 1$. Assume also that $x \to A(x)$ is non-negative definite and bounded and, moreover, that $x \to \sqrt{A(x)}$ is Lipschitz continuous in Ω. Then,

$$|Du(x)| \leq \frac{K}{d_{\partial\Omega}^{\frac{1}{p-1}}(x)} \quad \text{a.e. in } \ \Omega\,,$$

where $K > 0$ is a constant depending only on p, $\|A\|_{L^\infty(\Omega)}$, the Lipschitz constant of $\sqrt{A}$, and $\|d_{\partial\Omega}^{\frac{p}{p-1}}(f^+ + \lambda u^-)\|_{L^\infty(\Omega)}$.

Remark 3.2. In the completely degenerate case $A \equiv O$, one recovers the Lipschitz regularity which is known to hold for subsolutions of coercive first-order equations of the form

$$|Du|^p + \lambda u = f(x)$$

with $p > 1$, see, e.g., [1, 2, 14].

Remark 3.3. The result of Theorem 3.1 can be found in [16] for the equation

$$-\Delta u + |Du|^p + \lambda\, u = f(x)\,, \qquad x \in \Omega\,,$$

and in [3] for more general equations. The proof devised in [7] is not based, however, on the Bernstein method but rather on the tools introduced by H. Ishii and P.L. Lions [11] which do not require differentiation of the equation, allowing therefore some x dependent nonlinearities. Observe also that in this way an estimate not depending on $\|u\|_\infty$, but only on $\|\lambda\, u\|_\infty$ can be obtained. As pointed out above this a very useful fact in the analysis of homogenization problems.

Remark 3.4. The result above can be extended to more general equations having a first-order term of the form $H(x,\xi)$ with $H : \Omega \times \mathbb{R}^n \to R$ continuous and satisfying

$$H(x,\xi) \geq \gamma_0\, |\xi|^p\,, \quad p > 1\,,$$
$$|H(x,\xi) - H(x,\eta)| \leq \gamma_1\, (|\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|\,,$$
$$|H(x,\xi) - H(y,\xi)| \leq \gamma_2\, (g_1(x) + g_2(x)|\xi|^p)\, |x - y|\,.$$

The coercivity of $H(x,\cdot)$ is almost necessary for the validity of the result of Theorem 3.1 in the presence of degeneracies of the principal part.

4. The Dirichlet problem

It is well known from the fundamental works of Fichera [9] and Oleinik-Radkevic [17] that the degeneracy of the principal part of the equation is an obstruction to the solvability of the linear Dirichlet problem

$$\begin{cases} -\mathrm{Tr}\left(A(x)D^2u\right) = f(x) & \text{in } \Omega\,, \\ u = g & \text{on } \partial\Omega\,, \end{cases}$$

in the classical sense for arbitrarily prescribed boundary data. In the quasilinear model,

$$-\mathrm{Tr}\left(A(x)D^2u\right) + |Du|^p + \lambda\, u = f \quad \text{in } \Omega$$

the presence of the strongly quasilinear term $|Du|^p$ with $p > 2$ is another source of obstruction, even in the simplest case $A(x) \equiv I$. Indeed, for the Dirichlet problem

$$\begin{cases} -\Delta u + |Du|^p + \lambda\, u = f(x) & \text{in } \Omega\,, \\ u = g & \text{on } \partial\Omega \end{cases}$$

examples show that boundary layers can actually occur if $p > 2$, see [12].

From the analysis in [12, 4] it is known in fact that the best one can obtain in general is the existence of a viscosity solution $u \in C(\overline{\Omega})$ satisfying the boundary condition in the relaxed sense:

$$\begin{cases} -\Delta u + |Du|^p + \lambda\, u = f(x) & \text{in } \Omega\,, \\ \max\left[u(x) - g(x)\,;\, -\Delta u + |Du|^p + \lambda\, u - f(x)\right] = 0 & \text{on } \partial\Omega\,. \end{cases}$$

By our previous result Theorem 2.1, we know that for $p > 2$, every function $u \in C(\overline{\Omega})$ satisfying the equation in Ω is $\frac{p-2}{p-1}$-Hölder continuous up to $\partial\Omega$. Hence, a *necessary condition* in order that the solution can attain continuously the boundary data g is the existence of some $M \geq 0$ such that

$$|g(x) - g(y)| \leq M\, |x - y|^{\frac{p-2}{p-1}} \qquad \forall\, x,\, y \in \partial\Omega\,.$$

In the next result some *sufficient conditions* on the data ensuring the existence of a solution $u \in C(\overline{\Omega})$ of the Dirichlet problem

$$\begin{cases} -\mathrm{Tr}\left(A(x)D^2u\right) + |Du|^p + \lambda u = f(x) & \text{in } \Omega\,, \\ u(x) = g(x) & \text{on } \partial\Omega\,. \end{cases} \tag{4.1}$$

assuming the boundary datum at all points are exhibited. In view of the above remark, we assume then that

$$|g(x) - g(y)| \leq M|x - y|^{\frac{p-2}{p-1}} \quad \text{for all} \quad x, y \in \partial\Omega\,. \tag{4.2}$$

We assume also that Ω is open bounded with C^2 boundary and denote by ν the normal inward unit vector to $\partial\Omega$.

Theorem 4.1. *Assume that $x \to A(x)$ is non-negative definite and bounded and that $x \to \sqrt{A(x)}$ is Lipschitz continuous. Assume moreover $\lambda > 0$, $p > 2$ and that, for some $\sigma > 0$,*

$$A(x)\,\nu(x)\cdot\nu(x) \geq \sigma > 0 \qquad \forall x \in \partial\Omega\,. \tag{4.3}$$

Then there exists $M_0 > 0$ depending on p, n, σ, A, and Ω such that if g satisfies (4.2) *with $M < M_0$ and*

$$\lambda \inf_{\partial\Omega} g \leq \inf_{\overline{\Omega}} f$$

then the Dirichlet problem (4.1) *has a unique viscosity solution $u \in C^{0,\frac{p-2}{p-1}}(\overline{\Omega})$.*

Remark 4.2. Condition (4.3) is a non degeneracy condition at the boundary which is trivially satisfied if A is positive definite. In this case, the existence and uniqueness of solutions for (4.1) is well known, see [13].

For the proof, let us observe preliminarily that it is enough to prove the existence of a viscosity subsolution $v \in C(\overline{\Omega})$ satisfying $v = g$ on $\partial\Omega$. Indeed, by the results in [4] mentioned above, there exists $u \in C(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta u + |Du|^p + \lambda\, u = f(x) & \text{in } \Omega\,, \\ \max\left[u(x) - g(x)\,;\, -\Delta u + |Du|^p + \lambda\, u - f(x)\right] = 0 & \text{on } \partial\Omega\,. \end{cases}$$

If such a subsolution v exists, it can be proved that $u \geq v$ in $\overline{\Omega}$ and $u \leq g$ on $\partial\Omega$, so that $g = v \leq u \leq g$ on $\partial\Omega$.

The construction of the subsolution v is technically quite involved and we refer to [7] for all details. Let us only indicate here that the first step in the construction is to consider, for each fixed $y \in \partial\Omega$ the functions

$$v_y(x) = g(y) - M\,|x-y|^{\frac{p-2}{p-1}} - \mu\, M\, d_{\partial\Omega}^{\frac{p-2}{p-1}}(x)$$

where $\mu > 0$ is a parameter. A careful computation which makes use on the smoothness of $\partial\Omega$, the compatibility restriction between f and g and of condition (4.3), shows that each v_y is a C^2 subsolution of (4.1) in neighborhood of $\partial\Omega$.

By an important stability property of viscosity subsolutions, the function

$$v(x) = \sup_{y\in\partial\Omega} v_y(x)$$

is a viscosity subsolution as well.

The proof is concluded by a suitable extension procedure, requiring an appropriately large choice of the parameter μ, of function v to the whole domain Ω.

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Criteria for the L^p-dissipativity of Partial Differential Operators

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Dedicated to Vladimir Maz'ya on the occasion of his 70*th birthday*

Abstract. In this paper we present a survey of some results concerning the L^p-dissipativity of partial differential operators. These results were obtained in joint works with V. Maz'ya

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1. Introduction

The aim of the paper is to survey some results obtained with V. Maz'ya [4, 5] on the L^p-dissipativity of partial differential operators.

A linear operator $A : D(A) \subset L^p(\Omega) \to L^p(\Omega)$ (Ω being a domain of $\mathbb{R}^n$, $1 < p < \infty$) is said to be L^p-*dissipative* if

$$\mathscr{R}e \int_\Omega \langle Au, u\rangle |u|^{p-2} dx \leqslant 0 \tag{1.1}$$

for any $u \in D(A)$. Here and in the sequel the integrand is extended by zero on the set where u vanishes.

The dissipativity of the partial differential operator A and the corresponding contractivity of the semigroup generated by A have been widely investigated in many papers (see, e.g., [21, 3, 7, 1, 28, 8, 14, 26, 15, 9, 10, 18, 19, 17, 16, 2, 6, 13, 27, 20, 24, 25, 22]).

In particular it is well known that scalar second-order elliptic operators with real coefficients may generate contractive semigroup in L^p, and this was noticed for the first time in [21]. The case $p = \infty$ was considered in [15], where necessary and sufficient conditions for the L^∞-contractivity for scalar second-order strongly elliptic systems with smooth coefficients were given. Scalar second-order elliptic operators with complex coefficients were handled as a particular case. Necessary and sufficient conditions for the L^∞-contractivity were later given in [2] under the assumption that the coefficients are measurable and bounded.

The paper [4] is devoted to the Dirichlet problem for scalar second-order differential operators whose coefficients are complex measures. The main result is an algebraic necessary and sufficient condition for the L^p-dissipativity of the operator $\nabla^t(\mathscr{A}\,\nabla)$, where the matrix $\mathscr{I}m\,\mathscr{A}$ is symmetric. Several other results are obtained also for the more general operator $\nabla^t(\mathscr{A}\,\nabla u)+\mathbf{b}\nabla u+\nabla^t(\mathbf{c}u)+au$. A necessary and sufficient condition for the L^p-dissipativity of such an operator is obtained in the case of constant coefficients.

The problem of determining the angle of dissipativity of the operator $\nabla^t(\mathscr{A}\nabla)$ is solved in [5]. In this paper the L^p-dissipativity of systems of partial differential operators is also considered. In particular, a necessary and sufficient condition for the L^p-dissipativity of the two-dimensional elasticity is obtained.

2. A basic lemma and some consequences

Let us denote by $C_0(\Omega)$ the space of complex-valued continuous functions with compact support contained in Ω, where $\Omega\subset\mathbb{R}^n$ is supposed to be a domain.

Let $C_0^1(\Omega)$ be the space $C^1(\Omega)\cap C_0(\Omega)$.

By $\mathscr{A}$ we denote a $n\times n$ matrix, whose entries are complex-valued measures a^{hk} belonging to $(C_0(\Omega))^*$.

Let $\mathbf{b}=(b_1,\dots,b_n)$ and $\mathbf{c}=(c_1,\dots,c_n)$ stand for complex-valued vectors with $b_j,c_j\in(C_0(\Omega))^*$. By a we mean a complex-valued scalar distribution in $(C_0^1(\Omega))^*$.

Let $\mathscr{L}(u,v)$ be the sesquilinear form

$$\mathscr{L}(u,v)=\int_\Omega(\langle\mathscr{A}\,\nabla u,\nabla v\rangle-\langle\mathbf{b}\nabla u,v\rangle+\langle u,\overline{\mathbf{c}}\nabla v\rangle-a\langle u,v\rangle)$$

defined on $C_0^1(\Omega)\times C_0^1(\Omega)$.

We note that the form $\mathscr{L}$ is related to the operator

$$Au=\nabla^t(\mathscr{A}\,\nabla u)+\mathbf{b}\nabla u+\nabla^t(\mathbf{c}u)+au \tag{2.1}$$

where ∇^t denotes the divergence operator. The operator A acts from $C_0^1(\Omega)$ to $(C_0^1(\Omega))^*$ through the relation

$$\mathscr{L}(u,v)=-\int_\Omega\langle Au,v\rangle$$

for any $u,v\in C_0^1(\Omega)$. The integration has to be understood in the sense of distributions.

In [4] the following definition was given:

Definition 2.1. Let $1 < p < \infty$. The form $\mathscr{L}$ is called *L^p-dissipative* if for all $u \in C_0^1(\Omega)$

$$\mathscr{R}e\, \mathscr{L}(u, |u|^{p-2}u) \geqslant 0 \qquad \text{if } p \geqslant 2;$$
$$\mathscr{R}e\, \mathscr{L}(|u|^{p'-2}u, u) \geqslant 0 \qquad \text{if } 1 < p < 2$$

where $p' = p/(p-1)$ (we use here that $|u|^{q-2}u \in C_0^1(\Omega)$ for $q \geqslant 2$ and $u \in C_0^1(\Omega)$).

The following lemma is a basic one and it provides a necessary and sufficient condition for the L^p-dissipativity of the form $\mathscr{L}$:

Lemma 2.2 ([4])**.** *The form $\mathscr{L}$ is L^p-dissipative if and only if for all $v \in C_0^1(\Omega)$*

$$\mathscr{R}e \int_\Omega \Big[\langle \mathscr{A}\, \nabla v, \nabla v\rangle - (1-2/p)\langle(\mathscr{A} - \mathscr{A}^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle$$
$$-(1-2/p)^2\langle \mathscr{A}\, \nabla(|v|), \nabla(|v|)\rangle\Big]$$
$$+\int_\Omega \langle \mathscr{I}m\,(\mathbf{b}+\mathbf{c}), \mathscr{I}m\,(\overline{v}\nabla v)\rangle + \int_\Omega \mathscr{R}e\,(\nabla^t(\mathbf{b}/p - \mathbf{c}/p') - a)|v|^2 \geqslant 0.$$

This lemma has several interesting consequences. The first one is a necessary condition:

Corollary 2.3 ([4])**.** *If the form $\mathscr{L}$ is L^p-dissipative, we have*

$$\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle \geqslant 0$$

for any $\xi \in \mathbb{R}^n$.

It is easy to see that this condition is not sufficient for the L^p-dissipativity. On the other hand, Lemma 2.2 implies the following sufficient condition:

Corollary 2.4 ([4])**.** *Let α, β two real constants. If*

$$\frac{4}{p\,p'}\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle + \langle \mathscr{R}e\, \mathscr{A}\, \eta, \eta\rangle + 2\langle(p^{-1}\, \mathscr{I}m\, \mathscr{A} + p'^{-1}\, \mathscr{I}m\, \mathscr{A}^*)\xi, \eta\rangle$$
$$+\langle \mathscr{I}m\,(\mathbf{b}+\mathbf{c}), \eta\rangle - 2\langle \mathscr{R}e\,(\alpha\mathbf{b}/p - \beta\mathbf{c}/p'), \xi\rangle$$
$$+\mathscr{R}e\,\left[\nabla^t\left((1-\alpha)\mathbf{b}/p - (1-\beta)\mathbf{c}/p'\right) - a\right] \geqslant 0 \tag{2.2}$$

for any $\xi, \eta \in \mathbb{R}^n$, the form $\mathscr{L}$ is L^p-dissipative.

Note that, putting $\alpha = \beta = 0$ in (2.2), we find that if

$$\frac{4}{p\,p'}\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle + \langle \mathscr{R}e\, \mathscr{A}\, \eta, \eta\rangle + 2\langle(p^{-1}\, \mathscr{I}m\, \mathscr{A} + p'^{-1}\, \mathscr{I}m\, \mathscr{A}^*)\xi, \eta\rangle$$
$$+\langle \mathscr{I}m\,(\mathbf{b}+\mathbf{c}), \eta\rangle + \mathscr{R}e\,\left[\nabla^t\left(\mathbf{b}/p - \mathbf{c}/p'\right) - a\right] \geqslant 0 \tag{2.3}$$

for any $\xi, \eta \in \mathbb{R}^n$, the form $\mathscr{L}$ is L^p-dissipative.

Let us show that – generally speaking – the condition in Corollary 2.4 is not sufficient. Let $n = 2$ and

$$\mathscr{A} = \begin{pmatrix} 1 & i\gamma \\ -i\gamma & 1 \end{pmatrix}$$

where γ is a real constant, $\mathbf{b} = \mathbf{c} = a = 0$. In this case polynomial (2.3) is given by

$$(\eta_1 + \gamma\xi_2)^2 + (\eta_2 - \gamma\xi_1)^2 - (\gamma^2 - 4/(pp'))|\xi|^2.$$

Taking $\gamma^2 > 4/(pp')$, condition (2.3) is not satisfied, while we have the L^p-dissipativity, because the corresponding operator A is the Laplacian.

Note that here the matrix $\mathscr{I}m\, \mathscr{A}$ is not symmetric. Later we shall give another example in which A is L^p-dissipative and (2.3) is not satisfied, even if $\mathscr{I}m\, \mathscr{A}$ is symmetric (see below, after the Corollary 4.2).

The following corollaries are other consequences of Lemma 2.2.

Corollary 2.5 ([4]). *If the form $\mathscr{L}$ is both L^p- and $L^{p'}$-dissipative, it is also L^r-dissipative for any r between p and p', i.e., for any r given by*

$$1/r = t/p + (1-t)/p' \qquad (0 \leqslant t \leqslant 1). \tag{2.4}$$

Corollary 2.6 ([4]). *Suppose that either $\mathscr{I}m\, \mathscr{A} = 0, \mathscr{R}e\, \nabla^t \mathbf{b} = \mathscr{R}e\, \nabla^t \mathbf{c} = 0$ or $\mathscr{I}m\, \mathscr{A} = \mathscr{I}m\, \mathscr{A}^t, \mathscr{I}m\, (\mathbf{b}+\mathbf{c}) = 0, \mathscr{R}e\, \nabla^t \mathbf{b} = \mathscr{R}e\, \nabla^t \mathbf{c} = 0$. If $\mathscr{L}$ is L^p-dissipative, it is also L^r-dissipative for any r given by* (2.4).

3. The main result

Let us consider the operator (2.1) without lower-order terms:

$$Au = \nabla^t(\mathscr{A}\, \nabla u). \tag{3.1}$$

Like in (2.1), the coefficients a^{hk} are supposed to be in $(C_0(\Omega))^*$.

The following Theorem, which is the main result in [4], provides an algebraic necessary and sufficient condition for the L^p-dissipativity of operator (3.1).

Theorem 3.1. *Let the matrix $\mathscr{I}m\, \mathscr{A}$ be symmetric, i.e., $\mathscr{I}m\, \mathscr{A}^t = \mathscr{I}m\, \mathscr{A}$. The form*

$$\mathscr{L}(u,v) = \int_\Omega \langle \mathscr{A}\, \nabla u, \nabla v\rangle$$

is L^p-dissipative if and only if

$$|p-2|\, |\langle \mathscr{I}m\, \mathscr{A}\, \xi, \xi\rangle| \leqslant 2\sqrt{p-1}\, \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle \tag{3.2}$$

for any $\xi \in \mathbb{R}^n$, where $|\cdot|$ denotes the total variation.

It follows immediately from (3.2) that, if A is such that $\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle \geqslant 0$ for any $\xi \in \mathbb{R}^n$, then $\mathscr{L}$ is L^2-dissipative. If moreover A is a real coefficient operator, $\mathscr{L}$ is L^p-dissipative for any p.

Condition (3.2) is equivalent to the positiveness of a certain polynomial in ξ and η. Specifically (3.2) is equivalent to the condition

$$\frac{4}{p\,p'}\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle + \langle \mathscr{R}e\, \mathscr{A}\, \eta, \eta\rangle - 2(1-2/p)\langle \mathscr{I}m\, \mathscr{A}\, \xi, \eta\rangle \geqslant 0 \tag{3.3}$$

for any $\xi, \eta \in \mathbb{R}^n$.

Let us assume that either A has lower-order terms or they are absent and $\mathscr{I}m\, \mathscr{A}$ is not symmetric. One could prove that (3.2) is still a necessary condition for A to be L^p-dissipative. However, in general, it is not sufficient. This is shown by the next example (see also Theorem 4.1 below for the particular case of constant coefficients).

Let $n = 2$ and let Ω be a bounded domain. Denote by σ a not identically vanishing real function in $C_0^2(\Omega)$ and let $\lambda \in \mathbb{R}$. Consider operator (3.1) with

$$\mathscr{A} = \begin{pmatrix} 1 & i\lambda\partial_1(\sigma^2) \\ -i\lambda\partial_1(\sigma^2) & 1 \end{pmatrix},$$

i.e.,

$$Au = \partial_1(\partial_1 u + i\lambda\partial_1(\sigma^2)\,\partial_2 u) + \partial_2(-i\lambda\partial_1(\sigma^2)\,\partial_1 u + \partial_2 u),$$

where $\partial_i = \partial/\partial x_i$ $(i = 1, 2)$.

One can prove that, given σ, there exists $\lambda \in \mathbb{R}$ such that A is not L^2-dissipative, although condition (3.2) is satisfied.

Since A can be written as

$$Au = \Delta u - i\lambda(\partial_{21}(\sigma^2)\,\partial_1 u - \partial_{11}(\sigma^2)\,\partial_2 u),$$

the same example shows that (3.2) is not sufficient for the L^2-dissipativity in the presence of lower-order terms, even if $\mathscr{I}m\ \mathscr{A}$ is symmetric.

Let us consider now the operator (3.1) where $\mathscr{A} = \{a_{ij}(x)\}$ $(i, j = 1, \ldots, n)$ is a matrix with complex locally integrable entries defined in a domain $\Omega \subset \mathbb{R}^n$.

The next result provides a complete characterization of the angle of dissipativity of the operator A:

Theorem 3.2 ([5]). *Let the matrix $\mathscr{A}$ be symmetric. Let us suppose that the operator A is L^p-dissipative. Set*

$$\Lambda_1 = \operatorname*{ess\,inf}_{(x,\xi)\in\Xi} \frac{\langle \mathscr{I}m\ \mathscr{A}(x)\xi, \xi\rangle}{\langle \mathscr{R}e\ \mathscr{A}(x)\xi, \xi\rangle}, \quad \Lambda_2 = \operatorname*{ess\,sup}_{(x,\xi)\in\Xi} \frac{\langle \mathscr{I}m\ \mathscr{A}(x)\xi, \xi\rangle}{\langle \mathscr{R}e\ \mathscr{A}(x)\xi, \xi\rangle}$$

where

$$\Xi = \{(x, \xi) \in \Omega \times \mathbb{R}^n \mid \langle \mathscr{R}e\ \mathscr{A}(x)\xi, \xi\rangle > 0\}.$$

The operator zA is L^p-dissipative if and only if

$$\vartheta_- \leqslant \arg z \leqslant \vartheta_+\,,$$

where[1]

$$\vartheta_- = \begin{cases} \operatorname{arccot}\left(\frac{2\sqrt{p-1}}{|p-2|} - \frac{p^2}{|p-2|}\,\frac{1}{2\sqrt{p-1}+|p-2|\Lambda_1}\right) - \pi & \text{if } p \neq 2 \\ \operatorname{arccot}(\Lambda_1) - \pi & \text{if } p = 2 \end{cases}$$

$$\vartheta_+ = \begin{cases} \operatorname{arccot}\left(-\frac{2\sqrt{p-1}}{|p-2|} + \frac{p^2}{|p-2|}\,\frac{1}{2\sqrt{p-1}-|p-2|\Lambda_2}\right) & \text{if } p \neq 2 \\ \operatorname{arccot}(\Lambda_2) & \text{if } p = 2. \end{cases}$$

[1]Here $0 < \operatorname{arccot} y < \pi$, $\operatorname{arccot}(+\infty) = 0$, $\operatorname{arccot}(-\infty) = \pi$ and

$$\operatorname*{ess\,inf}_{(x,\xi)\in\Xi} \frac{\langle \mathscr{I}m\ \mathscr{A}(x)\xi, \xi\rangle}{\langle \mathscr{R}e\ \mathscr{A}(x)\xi, \xi\rangle} = +\infty, \quad \operatorname*{ess\,sup}_{(x,\xi)\in\Xi} \frac{\langle \mathscr{I}m\ \mathscr{A}(x)\xi, \xi\rangle}{\langle \mathscr{R}e\ \mathscr{A}(x)\xi, \xi\rangle} = -\infty$$

if Ξ has zero measure.

If $\mathscr{A}$ is a real matrix, then $\Lambda_1 = \Lambda_2 = 0$ and the angle of dissipativity does not depend on the operator. In fact we have

$$\frac{2\sqrt{p-1}}{|p-2|} - \frac{p^2}{2\sqrt{p-1}|p-2|} = -\frac{|p-2|}{2\sqrt{p-1}}$$

and Theorem 3.2 shows that zA is dissipative if and only if

$$\operatorname{arccot}\left(-\frac{|p-2|}{2\sqrt{p-1}}\right) - \pi \leqslant \arg z \leqslant \operatorname{arccot}\left(\frac{|p-2|}{2\sqrt{p-1}}\right),$$

i.e.,

$$|\arg z| \leqslant \arctan\left(\frac{2\sqrt{p-1}}{|p-2|}\right).$$

This is a well-known result (see, e.g., [11], [12], [23]).

4. The constant coefficients case

Generally speaking, it is not possible to give necessary and sufficient algebraic conditions for the L^p-dissipativity of the general operator (2.1). Consider for example the operator

$$Au = \Delta u + a(x)u$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. Denote by λ_1 the first eigenvalue of the Dirichlet problem for Laplace equation in Ω. A sufficient condition for A to be L^2-dissipative is $\mathscr{R}e\ a \leqslant \lambda_1$ and we cannot give an algebraic characterization of λ_1.

However it is still possible to find such conditions, provided the coefficients of the operator are constant.

Let A be the operator

$$Au = \nabla^t(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + au \tag{4.1}$$

with constant complex coefficients. Without loss of generality we can assume that the matrix $\mathscr{A}$ is symmetric.

The following theorem provides a necessary and sufficient condition for the L^p-dissipativity of the operator A.

Theorem 4.1 ([4]). *Let Ω be an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. The operator (4.1) is L^p-dissipative if and only if there exists a real constant vector V such that*

$$2\,\mathscr{R}e\,\mathscr{A}\,V + \mathscr{I}m\,\mathbf{b} = 0$$
$$\mathscr{R}e\ a + \langle \mathscr{R}e\,\mathscr{A}\,V, V\rangle \leqslant 0$$

and the inequality

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle \tag{4.2}$$

holds for any $\xi \in \mathbb{R}^n$.

If the matrix $\mathscr{R}e\ \mathscr{A}$ is not singular, we find

Corollary 4.2 ([4]). *Let Ω be an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. Let us suppose that the matrix $\mathscr{R}e\ \mathscr{A}$ is not singular. The operator A is L^p-dissipative if and only if* (4.2) *holds and*

$$4\,\mathscr{R}e\ a \leqslant -\langle (\mathscr{R}e\ \mathscr{A})^{-1}\,\mathscr{I}m\ \mathbf{b}, \mathscr{I}m\ \mathbf{b}\rangle. \tag{4.3}$$

We are now in a position to show that condition (2.2) is not necessary for the L^p-dissipativity, even if the matrix $\mathscr{I}m\ \mathscr{A}$ is symmetric.

Let $n = 1$ and $\Omega = \mathbb{R}^1$. Consider the operator

$$\left(1 + 2\,\frac{\sqrt{p-1}}{p-2}\,i\right) u'' + 2iu' - u,$$

where $p \neq 2$ is fixed. Conditions (4.2) and (4.3) are satisfied and this operator is L^p-dissipative, in view of Corollary 4.2.

On the other hand, the polynomial in (2.3) is

$$\left(2\,\frac{\sqrt{p-1}}{p}\,\xi - \eta\right)^2 + 2\eta + 1$$

which is not nonnegative for any $\xi, \eta \in \mathbb{R}$.

5. Smooth coefficients

In this section we consider the operator

$$Au = \nabla^t(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + a\,u \tag{5.1}$$

with the coefficients $a^{hk}, b^h \in C^1(\overline{\Omega})$, $a \in C^0(\overline{\Omega})$. Here Ω is a bounded domain in $\mathbb{R}^n$, whose boundary is in the class $C^{2,\alpha}$ for some $\alpha \in [0,1)$.

We consider A as an operator defined on the set

$$D(A) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega).$$

For such operators we can compare the classical concept of L^p-dissipativity of the operator A (see (1.1)) with the L^p-dissipativity of the corresponding form $\mathscr{L}$, as considered in the previous sections.

Theorem 5.1 ([4]). *The operator A is L^p-dissipative if and only if the form $\mathscr{L}$ is L^p-dissipative.*

If the operator A has smooth coefficients and no lower-order terms, it is possible to determine the best interval of p's for which the operator A is L^p-dissipative.

Define

$$\lambda = \inf_{(\xi,x)\in\mathscr{M}} \frac{\langle \mathscr{R}e\ \mathscr{A}(x)\xi, \xi\rangle}{|\langle \mathscr{I}m\ \mathscr{A}(x)\xi, \xi\rangle|}$$

where $\mathscr{M}$ is the set of (ξ, x) with $\xi \in \mathbb{R}^n$, $x \in \Omega$ such that $\langle \mathscr{I}m\ \mathscr{A}(x)\xi, \xi\rangle \neq 0$.

Theorem 5.2 ([4]). *Let A be the operator*

$$Au = \nabla^t(\mathscr{A}\,\nabla u). \tag{5.2}$$

Let us suppose that the matrix $\mathscr{I}m\ \mathscr{A}$ is symmetric and that

$$\langle \mathscr{R}e\ \mathscr{A}(x)\xi, \xi\rangle \geqslant 0$$

for any $x \in \Omega$, $\xi \in \mathbb{R}^n$. If $\mathscr{I}m\ \mathscr{A}(x) = 0$ for any $x \in \Omega$, A is L^p-dissipative for any $p > 1$. If $\mathscr{I}m\ \mathscr{A}$ does not vanish identically on Ω, A is L^p-dissipative if and only if

$$2 + 2\lambda(\lambda - \sqrt{\lambda^2+1}) \leqslant p \leqslant 2 + 2\lambda(\lambda + \sqrt{\lambda^2+1}).$$

This result implies a characterization of operators which are L^p-dissipative only for $p = 2$.

Corollary 5.3 ([4]). *Let A be as in Theorem 5.2. The operator A is L^p-dissipative only for $p = 2$ if and only if $\mathscr{I}m\ \mathscr{A}$ does not vanish identically and $\lambda = 0$.*

Other results can be proved if we suppose that the operator A is strongly elliptic in Ω in the sense that

$$\langle \mathscr{R}e\ \mathscr{A}(x)\xi, \xi\rangle > 0$$

for any $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n \setminus \{0\}$.

In this case one can prove the contractivity of the semigroup generated by A.

Theorem 5.4 ([4]). *Let A be the strongly elliptic operator (5.2) with $\mathscr{I}m\ \mathscr{A} = \mathscr{I}m\ \mathscr{A}^t$. The operator A generates a contraction semigroup on L^p if and only if*

$$|p-2|\,|\langle \mathscr{I}m\ \mathscr{A}(x)\xi, \xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\ \mathscr{A}(x)\xi, \xi\rangle \tag{5.3}$$

for any $x \in \Omega$, $\xi \in \mathbb{R}^n$.

We know that condition (5.3) is necessary and sufficient for the L^p-dissipativity of the operator (5.2), provided $\mathscr{I}m\ \mathscr{A}$ is symmetric and there are no lower-order terms. We have shown that this is not true for the more general operator (5.1). The next result shows that condition (5.3) is necessary and sufficient for the L^p-quasi-dissipativity of (5.1). This means that there exists $\omega \geqslant 0$ such that $A - \omega I$ is L^p-dissipative, i.e.,

$$\mathscr{R}e \int_\Omega \langle Au, u\rangle |u|^{p-2} dx \leqslant \omega\, \|u\|_p^p$$

for any $u \in D(A)$.

We emphasize that here we do not require the symmetry of $\mathscr{I}m\ \mathscr{A}$.

Theorem 5.5 ([4]). *The strongly elliptic operator (5.1) is L^p-quasi-dissipative if and only if*

$$|p-2|\,|\langle \mathscr{I}m\ \mathscr{A}(x)\xi, \xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\ \mathscr{A}(x)\xi, \xi\rangle \tag{5.4}$$

for any $x \in \Omega$, $\xi \in \mathbb{R}^n$.

Recalling that A generates a quasi-contraction semigroup on L^p if there exists $\omega \geqslant 0$ such that $A - \omega I$ generates a contraction semigroup, we have also:

Theorem 5.6 ([4]). *Let A be the strongly elliptic operator* (5.1). *The operator A generates a quasi-contraction semigroup on L^p if and only if* (5.4) *holds for any $x \in \Omega$, $\xi \in \mathbb{R}^n$.*

6. The two-dimensional elasticity

Let us consider the classical operator of two-dimensional elasticity

$$Eu = \Delta u + (1-2\nu)^{-1}\nabla \quad {}^t u \tag{6.1}$$

where ν is the Poisson ratio. It is well known that E is strongly elliptic if and only if either $\nu > 1$ or $\nu < 1/2$.

In order to obtain some conditions for the L^p-dissipativity of the elasticity system, we see at first some results concerning systems of partial differential equations of the form

$$A = \partial_h(\mathscr{A}^{hk}(x)\partial_k) \tag{6.2}$$

where $\mathscr{A}^{hk}(x) = \{a_{ij}^{hk}(x)\}$ are $m \times m$ matrices whose elements are complex locally integrable functions defined in an arbitrary domain Ω of $\mathbb{R}^n$ $(1 \leqslant i,j \leqslant m,\ 1 \leqslant h,k \leqslant n)$.

Lemma 6.1 ([5]). *The operator* (6.2) *is L^p-dissipative in the domain $\Omega \subset \mathbb{R}^n$ if and only if*

$$\int_\Omega \Big(\mathscr{R}e\, \langle \mathscr{A}^{hk}\, \partial_k v, \partial_h v\rangle - (1-2/p)^2 |v|^{-4}\, \mathscr{R}e\, \langle \mathscr{A}^{hk}\, v, v\rangle\, \mathscr{R}e\, \langle v, \partial_k v\rangle\, \mathscr{R}e\, \langle v, \partial_h v\rangle$$
$$- (1-2/p)|v|^{-2}\, \mathscr{R}e\, (\langle \mathscr{A}^{hk}\, v, \partial_h v\rangle\, \mathscr{R}e\, \langle v, \partial_k v\rangle - \langle \mathscr{A}^{hk}\, \partial_k v, v\rangle\, \mathscr{R}e\, \langle v, \partial_h v\rangle)\Big) dx \geqslant 0$$

for any $v \in (C_0^1(\Omega))^m$.

The previous lemma holds in any number of variables. In the particular case $n = 2$ we have also:

Theorem 6.2 ([5]). *Let Ω be a domain of $\mathbb{R}^2$. If the operator* (6.2) *is L^p-dissipative, we have*

$$\begin{aligned}&\mathscr{R}e\, \langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\lambda, \lambda\rangle - (1-2/p)^2\, \mathscr{R}e\, \langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega, \omega\rangle (\mathscr{R}e\, \langle \lambda, \omega\rangle)^2\\ &-(1-2/p)\, \mathscr{R}e\, (\langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega, \lambda\rangle - \langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\lambda, \omega\rangle)\, \mathscr{R}e\, \langle \lambda, \omega\rangle \geqslant 0\end{aligned}$$

for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^2$, λ, $\omega \in \mathbb{C}^m$, $|\omega| = 1$.

By means of Lemma 6.1 and Theorem 6.2 it is possible to prove the following necessary and sufficient condition for the L^p-dissipativity of the two-dimensional elasticity:

Theorem 6.3 ([5]). *The operator* (6.1) *is L^p-dissipative if and only if*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}.$$

From this result one has the possibility of making a comparison between E and Δ from the point of view of the L^p-dissipativity.

Corollary 6.4 ([5]). *There exists $k>0$ such that $E-k\Delta$ is L^p-dissipative if and only if*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 < \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,.$$

There exists $k<2$ such that $k\Delta-E$ is L^p-dissipative if and only if

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 < \frac{2\nu(2\nu-1)}{(1-4\nu)^2}\,.$$

7. Systems of ordinary differential equations

Some necessary and sufficient conditions have been obtained for systems of ordinary differential operators. They concern the operator

$$Au=(\mathscr{A}(x)u')' \tag{7.1}$$

where $\mathscr{A}(x)=\{a_{ij}(x)\}$ $(i,j=1,\dots,m)$ is a matrix with complex locally integrable entries defined in the bounded or unbounded interval (a,b).

The corresponding sesquilinear form $\mathscr{L}(u,v)$ is given by

$$\mathscr{L}(u,v)=\int_a^b \langle \mathscr{A}\,u',v'\rangle\,dx.$$

Theorem 7.1 ([5]). *The operator A is L^p-dissipative if and only if*

$$\begin{aligned}&\mathscr{R}e\,\langle\mathscr{A}(x)\lambda,\lambda\rangle-(1-2/p)^2\,\mathscr{R}e\,\langle\mathscr{A}(x)\omega,\omega\rangle(\mathscr{R}e\,\langle\lambda,\omega\rangle)^2\\ &-(1-2/p)\,\mathscr{R}e\,(\langle\mathscr{A}(x)\omega,\lambda\rangle-\langle\mathscr{A}(x)\lambda,\omega\rangle)\,\mathscr{R}e\,\langle\lambda,\omega\rangle\geqslant 0\end{aligned}$$

for almost every $x\in(a,b)$ and for any $\lambda,\omega\in\mathbb{C}^m$, $|\omega|=1$.

A consequence of this theorem is

Corollary 7.2 ([5]). *If the operator A is L^p-dissipative, then*

$$\mathscr{R}e\,\langle\mathscr{A}(x)\lambda,\lambda\rangle\geqslant 0$$

for almost every $x\in(a,b)$ and for any $\lambda\in\mathbb{C}^m$.

We can precisely determine the angle of dissipativity of operator (7.1) with complex coefficients.

Theorem 7.3 ([5]). *Let the operator* (7.1) *be L^p-dissipative. The operator zA is L^p-dissipative if and only if*

$$\vartheta_- \leqslant \arg z \leqslant \vartheta_+$$

where

$$\vartheta_- = \operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\lambda,\omega)\in\Xi}(Q(x,\lambda,\omega)/P(x,\lambda,\omega))\right) - \pi,$$

$$\vartheta_+ = \operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\lambda,\omega)\in\Xi}(Q(x,\lambda,\omega)/P(x,\lambda,\omega))\right),$$

$$\begin{aligned}P(x,\lambda,\omega) = {}&\mathscr{R}e\,\langle\mathscr{A}(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathscr{R}e\,\langle\mathscr{A}(x)\omega,\omega\rangle(\mathscr{R}e\,\langle\lambda,\omega\rangle)^2\\ &- (1-2/p)\,\mathscr{R}e\,(\langle\mathscr{A}(x)\omega,\lambda\rangle - \langle\mathscr{A}(x)\lambda,\omega\rangle)\,\mathscr{R}e\,\langle\lambda,\omega\rangle,\\ Q(x,\lambda,\omega) = {}&\mathscr{I}m\,\langle\mathscr{A}(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathscr{I}m\,\langle\mathscr{A}(x)\omega,\omega\rangle(\mathscr{R}e\,\langle\lambda,\omega\rangle)^2\\ &- (1-2/p)\,\mathscr{I}m\,(\langle\mathscr{A}(x)\omega,\lambda\rangle - \langle\mathscr{A}(x)\lambda,\omega\rangle)\,\mathscr{R}e\,\langle\lambda,\omega\rangle\end{aligned}$$

and Ξ is the set

$$\Xi = \{(x,\lambda,\omega) \in (a,b)\times\mathbb{C}^m\times\mathbb{C}^m \mid |\omega| = 1,\ P^2(x,\lambda,\omega) + Q^2(x,\lambda,\omega) > 0\}.$$

Another consequence of Theorem 7.1 is the possibility of making a comparison between A and the operator $I(d^2/dx^2)$.

Corollary 7.4 ([5]). *There exists $k > 0$ such that $A - kI(d^2/dx^2)$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,inf}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m\\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) > 0.$$

There exists $k > 0$ such that $kI(d^2/dx^2) - A$ is L^p-dissipative if and only if

$$\operatorname*{ess\,sup}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m\\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) < \infty.$$

There exists $k \in \mathbb{R}$ such that $A - kI(d^2/dx^2)$ is L^p-dissipative if and only if

$$\operatorname*{ess\,inf}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m\\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) > -\infty.$$

In the particular case in which the coefficients a_{ij} of operator (7.1) are real, we can give a necessary and sufficient condition for the L^p-dissipativity of A.

Theorem 7.5 ([5]). *Let $\mathscr{A}$ be a real matrix $\{a_{hk}\}$ with $h,k = 1,\dots,m$. Let us suppose $\mathscr{A} = \mathscr{A}^t$ and $\mathscr{A} \geqslant 0$ (in the sense $\langle\mathscr{A}(x)\xi,\xi\rangle \geqslant 0$, for almost every $x \in (a,b)$ and for any $\xi \in \mathbb{R}^m$). The operator A is L^p-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1(x) + \mu_m(x))^2 \leqslant \mu_1(x)\mu_m(x)$$

almost everywhere, where $\mu_1(x)$ and $\mu_m(x)$ are the smallest and the largest eigenvalues of the matrix $\mathscr{A}(x)$ respectively. In the particular case $m = 2$ this condition *is equivalent to*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 (\operatorname{tr}\mathscr{A}(x))^2 \leqslant \det \mathscr{A}(x)$$

almost everywhere.

We have also:

Corollary 7.6 ([5]). *Let $\mathscr{A}$ be a real and symmetric matrix. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k>0$ such that $A - kI(d^2/dx^2)$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,inf}_{x\in(a,b)} \Big[(1+\sqrt{p\,p'}/2)\,\mu_1(x) + (1-\sqrt{p\,p'}/2)\,\mu_m(x)\Big] > 0. \tag{7.2}$$

In the particular case $m=2$ condition (7.2) is equivalent to

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[\operatorname{tr}\mathscr{A}(x) - \frac{\sqrt{p\,p'}}{2}\sqrt{(\operatorname{tr}\mathscr{A}(x))^2 - 4\det\mathscr{A}(x)}\right] > 0.$$

If we require something more about the matrix $\mathscr{A}$ we have also

Corollary 7.7 ([5]). *Let $\mathscr{A}$ be a real and symmetric matrix. Suppose $\mathscr{A} \geqslant 0$ almost everywhere. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. If there exists $k>0$ such that $A - kI(d^2/dx^2)$ is L^p-dissipative, then*

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[\mu_1(x)\mu_m(x) - \left(\frac{1}{2}-\frac{1}{p}\right)^2 (\mu_1(x)+\mu_m(x))^2\right] > 0. \tag{7.3}$$

If, in addition, there exists C such that

$$\langle \mathscr{A}(x)\xi,\xi\rangle \leqslant C|\xi|^2 \tag{7.4}$$

for almost every $x\in(a,b)$ and for any $\xi\in\mathbb{R}^m$, the converse is also true. In the particular case $m=2$ condition (7.3) is equivalent to

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[\det\mathscr{A}(x) - \left(\frac{1}{2}-\frac{1}{p}\right)^2 (\operatorname{tr}\mathscr{A}(x))^2\right] > 0.$$

Generally speaking, assumption (7.4) cannot be omitted even if $\mathscr{A}\geqslant 0$, as it is showed by an example given in [5].

Corollary 7.8 ([5]). *Let $\mathscr{A}$ be a real and symmetric matrix. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k>0$ such that $kI(d^2/dx^2) - A$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,sup}_{x\in(a,b)} \Big[(1-\sqrt{p\,p'}/2)\,\mu_1(x) + (1+\sqrt{p\,p'}/2)\,\mu_m(x)\Big] < \infty. \tag{7.5}$$

In the particular case $m=2$ condition (7.5) is equivalent to

$$\operatorname*{ess\,sup}_{x\in(a,b)} \left[\operatorname{tr}\mathscr{A}(x) + \frac{\sqrt{p\,p'}}{2}\sqrt{(\operatorname{tr}\mathscr{A}(x))^2 - 4\det\mathscr{A}(x)}\right] < \infty.$$

In the case of a positive matrix $\mathscr{A}$, we have

Corollary 7.9 ([5]). *Let $\mathscr{A}$ be a real and symmetric matrix. Suppose $\mathscr{A} \geqslant 0$ almost everywhere. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k > 0$ such that $kI(d^2/dx^2) - A$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,sup}_{x\in(a,b)} \mu_m(x) < \infty.$$

8. Systems of partial differential equations

The results of the previous section lead to necessary and sufficient conditions for the L^p-dissipativity of a system of partial differential operators of the form

$$Au = \partial_h(\mathscr{A}^h(x)\partial_h u) \tag{8.1}$$

where $\mathscr{A}^h(x) = \{a^h_{ij}(x)\}$ $(i, j = 1, \dots, m)$ are matrices with complex locally integrable entries defined in a domain $\Omega \subset \mathbb{R}^n$ $(h = 1, \dots, n)$.

By y_h we denote the $(n-1)$-dimensional vector $(x_1, \dots, x_{h-1}, x_{h+1}, \dots, x_n)$ and we set $\omega(y_h) = \{x_h \in \mathbb{R} \mid x \in \Omega\}$.

Lemma 8.1 ([5]). *The operator* (8.1) *is L^p-dissipative if and only if the ordinary differential operators*

$$A(y_h)[u(x_h)] = d(\mathscr{A}^h(x)du/dx_h)/dx_h$$

are L^p-dissipative in $\omega(y_h)$ for almost every $y_h \in \mathbb{R}^{n-1}$ $(h = 1, \dots, n)$. This condition is void if $\omega(y_h) = \emptyset$.

Theorem 8.2 ([5]). *The operator* (8.1) *is L^p-dissipative if and only if*

$$\begin{aligned}&\mathscr{R}e\,\langle\mathscr{A}^h(x_0)\lambda, \lambda\rangle - (1-2/p)^2\,\mathscr{R}e\,\langle\mathscr{A}^h(x_0)\omega, \omega\rangle(\mathscr{R}e\,\langle\lambda, \omega\rangle)^2\\ &-(1-2/p)\,\mathscr{R}e\,(\langle\mathscr{A}^h(x_0)\omega, \lambda\rangle - \langle\mathscr{A}^h(x_0)\lambda, \omega\rangle)\,\mathscr{R}e\,\langle\lambda, \omega\rangle \geqslant 0\end{aligned} \tag{8.2}$$

for almost every $x_0 \in \Omega$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \dots, n$.

It is interesting to remark that in the scalar case $(m = 1)$, operator (8.1) falls into the operators considered in Section 3. One can prove directly that in this case condition (8.2) is equivalent to (3.3) (see [5, p. 262]).

Theorem 8.2 permits to determine the angle of dissipativity of operator (8.1):

Theorem 8.3 ([5]). *Let A be L^p-dissipative. The operator zA is L^p-dissipative if and only if $\vartheta_- \leqslant \arg z \leqslant \vartheta_+$, where*

$$\vartheta_- = \max_{h=1,\dots,n} \operatorname{arccot}\Bigg(\operatorname*{ess\,inf}_{(x,\lambda,\omega)\in\Xi_h} (Q_h(x,\lambda,\omega)/P_h(x,\lambda,\omega))\Bigg) - \pi,$$

$$\vartheta_+ = \min_{h=1,\dots,n} \operatorname{arccot}\Bigg(\operatorname*{ess\,sup}_{(x,\lambda,\omega)\in\Xi_h} (Q_h(x,\lambda,\omega)/P_h(x,\lambda,\omega))\Bigg),$$

and

$$P_h(x,\lambda,\omega) = \mathscr{R}e\,\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathscr{R}e\,\langle \mathscr{A}^h(x)\omega,\omega\rangle(\mathscr{R}e\,\langle\lambda,\omega\rangle)^2 - (1-2/p)\,\mathscr{R}e\,(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\,\mathscr{R}e\,\langle\lambda,\omega\rangle,$$

$$Q_h(x,\lambda,\omega) = \mathscr{I}m\,\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathscr{I}m\,\langle \mathscr{A}^h(x)\omega,\omega\rangle(\mathscr{R}e\,\langle\lambda,\omega\rangle)^2 - (1-2/p)\,\mathscr{I}m\,(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\,\mathscr{R}e\,\langle\lambda,\omega\rangle,$$

$$\Xi_h = \{(x,\lambda,\omega)\in\Omega\times\mathbb{C}^m\times\mathbb{C}^m \mid |\omega|=1,\ P_h^2(x,\lambda,\omega)+Q_h^2(x,\lambda,\omega)>0\}.$$

If A has real coefficients, we can characterize the L^p-dissipativity in terms of the eigenvalues of the matrices $\mathscr{A}^h(x)$:

Theorem 8.4 ([5]). *Let A be the operator (8.1), where $\mathscr{A}^h$ are real matrices $\{a_{ij}^h\}$ with $i,j=1,\dots,m$. Let us suppose $\mathscr{A}^h=(\mathscr{A}^h)^t$ and $\mathscr{A}^h\geqslant 0$ ($h=1,\dots,n$). The operator A is L^p-dissipative if and only if*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2(\mu_1^h(x)+\mu_m^h(x))^2 \leqslant \mu_1^h(x)\,\mu_m^h(x)$$

for almost every $x\in\Omega$, $h=1,\dots,n$, where $\mu_1^h(x)$ and $\mu_m^h(x)$ are the smallest and the largest eigenvalues of the matrix $\mathscr{A}^h(x)$ respectively. In the particular case $m=2$ this condition is equivalent to

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2(\operatorname{tr}\mathscr{A}^h(x))^2 \leqslant \det\mathscr{A}^h(x)$$

for almost every $x\in\Omega$, $h=1,\dots,n$.

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Sharp Estimates for Nonlinear Potentials and Applications

Andrea Cianchi

To Vladimir Maz'ya, on the occasion of his seventieth birthday

Abstract. A sharp estimate for the decreasing rearrangement of the nonlinear potential of a function in terms of the rearrangement of the function itself is presented. As a consequence, boundedness properties of nonlinear potentials in rearrangement invariant spaces are characterized. In particular, the case of Orlicz and Lorentz spaces is discussed. Applications to rearrangement estimates for local solutions to quasilinear elliptic PDE's and for their gradients are given.

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1. Introduction

The classical Riesz potential of order $\alpha \in (0, n)$ of a locally integrable function f in $\mathbb{R}^n$, $n \geq 1$, is defined as

$$\mathbf{I}_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \qquad \text{for } x \in \mathbb{R}^n. \tag{1}$$

The operator I_α enters in various branches of analysis, including potential theory, harmonic analysis, elliptic partial differential equations, Sobolev spaces. In particular, if $n > 2$ and ω_n denotes the measure of the unit ball in $\mathbb{R}^n$, then the function $\frac{1}{n(n-2)\omega_n} I_2 f(x)$ is the Newtonian potential in $\mathbb{R}^n$, namely the unique solution decaying to 0 at infinity to the Poisson equation

$$-\Delta u = f \qquad \text{in } \mathbb{R}^n. \tag{2}$$

In certain problems involving the p-Laplace operator instead of the standard Laplace operator, a role is played by the so-called nonlinear potential $\mathbf{V}_{\alpha,p} f$, given for $p \in (1, \infty)$ and $\alpha \in (0, n)$ by

$$\mathbf{V}_{\alpha,p} f(x) = \mathbf{I}_\alpha (\mathbf{I}_\alpha |f|)^{\frac{1}{p-1}}(x) \qquad \text{for } x \in \mathbb{R}^n. \tag{3}$$

Introduced some forty years ago by V.P. Havin-V.G. Maz'ya [HM], and investigated by D.R. Adams-N. Meyers [AM] in the framework of nonlinear capacity theory, the nonlinear potentials $\mathbf{V}_{\alpha,p}$, and the closely related Wolff potentials, have seen a renewed interest in recent years in connection with the study of pointwise properties of solutions to nonlinear elliptic and parabolic partial differential equations [DM1, DM2, KM, L, TW].

The aim of the present note is to announce some results concerning estimates for the nonlinear potential operator $\mathbf{V}_{\alpha,p}$, and applications to elliptic partial differential equations. Their proofs and other results on this matter can be found in [C].

Our point of departure is a sharp estimate for the decreasing rearrangement of $\mathbf{V}_{\alpha,p} f$ in terms of the rearrangement of f. This enables us to discuss boundedness properties of $\mathbf{V}_{\alpha,p}$ in rearrangement invariant spaces, and in particular, in Orlicz and Lorentz spaces. A combination of these estimates for $\mathbf{V}_{\alpha,p}$ with results from [KM] and from the very recent contribution [DM2] yields bounds for solutions to quasilinear elliptic equations and for their gradient, both in rearrangement form and in the form of norm inequalities.

2. Estimates for potentials

Our basic result is a rearrangement inequality between $\mathbf{V}_{\alpha,p} f$ and f. Recall that, given a measurable set $\Omega \subset \mathbb{R}^n$ and a measurable function $f : \Omega \to \mathbb{R}$ such that

$$|\{x : |f(x)| > t\}| < \infty \qquad \text{for } t > 0, \tag{4}$$

the decreasing rearrangement of f is the function $f^* : [0, \infty) \to [0, \infty]$ defined as

$$f^*(s) = \sup\{t \geq 0 : |\{x \in \Omega : |f(x)| > t\}| > s\} \qquad \text{for } s \geq 0.$$

In other words, f^* is the (unique) non increasing, right-continuous function in $[0, \infty)$ equidistributed with f.

Theorem 2.1. *Let $p > 1$ and $0 < \alpha < \frac{n}{p}$. Then there exist constants $C = C(\alpha, p, n)$ and $k = k(\alpha, p, n)$ such that*

$$(\mathbf{V}_{\alpha,p} f)^*(s) \leq C \left[\left(s^{\frac{\alpha p}{n} - 1} \int_0^{ks} f^*(r) dr \right)^{\frac{1}{p-1}} + \int_{ks}^\infty r^{\frac{\alpha p'}{n} - 1} f^*(r)^{\frac{1}{p-1}} dr \right] \quad \text{for } s > 0, \tag{5}$$

for every measurable function f in $\mathbb{R}^n$ fulfilling (4). The result is sharp, in the sense that, for every $k > 0$, there exists a positive constant $C' = C'(\alpha, p, n, k)$

such that, if f is nonnegative and radially decreasing, then

$$(\mathbf{V}_{\alpha,p}f)^*(s) \geq C'\left[\left(s^{\frac{\alpha p}{n}-1}\int_0^{ks} f^*(r)dr\right)^{\frac{1}{p-1}} + \int_{ks}^{\infty} r^{\frac{\alpha p'}{n}-1} f^*(r)^{\frac{1}{p-1}}dr\right] \quad \text{for } s > 0. \tag{6}$$

Theorem 2.1 can be used to reduce the problem of the boundedness of the operator $\mathbf{V}_{\alpha,p}$ between arbitrary rearrangement invariant spaces on $\mathbb{R}^n$ to a pair of more elementary inequalities for one-dimensional Hardy type operators in their representation spaces. A rearrangement invariant space $X(\Omega)$ is a Banach function space (in the sense of Luxemburg) of real-valued measurable functions on Ω endowed with a norm $\|\cdot\|_{X(\Omega)}$ satisfying

$$\|f\|_{X(\Omega)} = \|g\|_{X(\Omega)} \qquad \text{if} \quad f^* = g^*. \tag{7}$$

If Ω' is a measurable subset of Ω and $\chi_{\Omega'}$ stands for the characteristic function of Ω', we set $\|f\|_{X(\Omega')} = \|f\chi_{\Omega'}\|_{X(\Omega)}$ for any measurable function f on Ω.

The representation space of an r.i. space $X(\Omega)$ is the r.i. space $\overline{X}(0,|\Omega|)$ having the property that

$$\|f\|_{X(\Omega)} = \|f^*\|_{\overline{X}(0,|\Omega|)} \tag{8}$$

for every $f \in X(\Omega)$. Note that, for customary spaces $X(\Omega)$, an expression for the norm $\|\cdot\|_{\overline{X}(0,|\Omega|)}$ is immediately derived from equation (8), via elementary properties of rearrangements.

Corollary 2.2. *Let $p > 1$ and $0 < \alpha < \frac{n}{p}$. Let $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ be rearrangement invariant spaces. Then there exists a constant C_1 such that*

$$\|(\mathbf{V}_{\alpha,p}f)^{p-1}\|_{Y(\mathbb{R}^n)} \leq C_1\|f\|_{X(\mathbb{R}^n)} \tag{9}$$

for every $f \in X(\mathbb{R}^n)$ if and only if there exists a constant C_2 such that

$$\left\|s^{\frac{\alpha p}{n}-1}\int_0^s \phi(r)dr\right\|_{\overline{Y}(0,\infty)} \leq C_2\|\phi\|_{\overline{X}(0,\infty)} \tag{10}$$

and

$$\left\|\left(\int_s^{\infty} r^{\frac{\alpha p'}{n}-1}\phi(r)^{\frac{1}{p-1}}dr\right)^{p-1}\right\|_{\overline{Y}(0,\infty)} \leq C_2\|\phi\|_{\overline{X}(0,\infty)} \tag{11}$$

for every nonnegative non-increasing function $\phi \in \overline{X}(0,\infty)$.

Lebesgue spaces are probably the most classical instance of r.i. spaces. Lorentz spaces and Orlicz spaces provide generalizations of Lebesgue spaces in different directions. Theorem 2.1 and Corollary 2.2 can be used, for example, to derive boundedness properties of nonlinear potentials in these spaces.

Recall that with any Young function A, namely a convex function from $[0,\infty)$ into $[0,\infty]$ vanishing at 0, it is associated the Orlicz space $L^A(\Omega)$, namely the r.i. space of those measurable functions f in Ω such that the Luxemburg norm

$$\|f\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A\Big(\frac{|f(x)|}{\lambda}\Big)\,dx \leq 1\right\}$$

is finite. Given Young functions A and B, define the functions $E, F : [0,\infty) \to [0,\infty]$ as

$$E(t) = \begin{cases} \left(\int_0^t \left(\frac{\tau^{\frac{1}{p-1} - 1 + \frac{\alpha p'}{n}}}{A(\tau)^{\frac{\alpha p'}{n}}} \right)^{\frac{n}{n-\alpha p'}} d\tau \right)^{p-1-\frac{\alpha p}{n}} & \text{if } \frac{n}{\alpha p'} > 1, \\ \sup_{\tau \in (0,t)} \frac{\tau}{A(\tau)^{\frac{\alpha p}{n}}} & \text{if } \frac{n}{\alpha p'} \le 1, \end{cases} \tag{12}$$

and

$$F(t) = \left(\int_0^t \frac{B(\tau)}{\tau^{1+\frac{n}{n-\alpha p}}} d\tau \right)^{\frac{n-\alpha p}{n}} \tag{13}$$

for $t > 0$. Then the following holds.

Theorem 2.3. *Let α, p and n be as in Theorem 2.1. Let A and B be Young functions such that the functions E and F given by (12) and (13) are finite valued, and there exists $\gamma > 0$ such that*

$$F\Big(\frac{E(s)}{\gamma}\Big) \le \gamma \frac{A(s)}{s} \qquad \text{for } s > 0. \tag{14}$$

Then there exists a constant $C = C(\alpha, \gamma, p, n)$ such that

$$\|(\mathbf{V}_{\alpha,p} f)^{p-1}\|_{L^B(\mathbb{R}^n)} \le C \|f\|_{L^A(\mathbb{R}^n)} \tag{15}$$

for every $f \in L^A(\mathbb{R}^n)$.

Note that a variant of Theorem 2.3 holds if Orlicz spaces on subsets of $\mathbb{R}^n$ having finite measure are taken into account, and condition (14) is fulfilled just for large values of s. Also, in this case the finiteness of the functions E and F is not a restriction, since A and B can be replaced, if necessary, by equivalent functions near 0 making E and F finite. Such a replacement leaves the relevant Orlicz spaces unchanged, up to equivalent norms.

Boundedness properties of $\mathbf{V}_{\alpha,p}$ in Lorentz spaces are described in Theorem 2.4 below. Recall that for $\sigma, \eta \in (0,\infty]$ the Lorentz space $L^{\sigma,\eta}(\Omega)$ is defined as the set of all measurable functions f in Ω for which the quantity

$$\|f\|_{L^{\sigma,\eta}(\Omega)} = \|s^{\frac{1}{\sigma} - \frac{1}{\eta}} f^*(s)\|_{L^\eta(0,|\Omega|)} \tag{16}$$

is finite. If $1 \le \eta \le \sigma$, such a quantity is actually a norm, and $L^{\sigma,\eta}(\Omega)$ is a rearrangement invariant space equipped with this norm. Otherwise, the expression on the right-hand side of (16) is a quasi-norm, in the sense that it fulfils the triangle inequality only up to a multiplicative constant.

Theorem 2.4. *Let α, p and n be as in Theorem 2.1.*

(i) *If $0 < \eta \le \infty$ and $1 < \sigma < \frac{n}{\alpha p}$, then there exists a constant $C = C(\alpha, n, p, \sigma, \eta)$ such that*

$$\|\mathbf{V}_{\alpha,p} f\|_{L^{\frac{\sigma n(p-1)}{n-\sigma\alpha p}, \eta(p-1)}(\mathbb{R}^n)} \le C \|f\|_{L^{\sigma,\eta}(\mathbb{R}^n)}^{\frac{1}{p-1}} \tag{17}$$

for every $f \in L^{\sigma,\eta}(\mathbb{R}^n)$.

(ii) *If* $\sigma = 1$ *and* $0 < \eta \leq 1$, *then there exists a constant* $C = C(\alpha, n, p, \eta)$ *such that*

$$\|\mathbf{V}_{\alpha,p} f\|_{L^{\frac{n(p-1)}{n-\alpha p},\infty}(\mathbb{R}^n)} \leq C \|f\|_{L^{1,\eta}(\mathbb{R}^n)}^{\frac{1}{p-1}} \tag{18}$$

for every $f \in L^{1,\eta}(\mathbb{R}^n)$.

(iii) *If* $\sigma = \frac{n}{\alpha p}$ *and* $\eta = \frac{1}{p-1}$, *then there exists a constant* $C = C(\alpha, n, p)$ *such that*

$$\|\mathbf{V}_{\alpha,p} f\|_{L^\infty(\mathbb{R}^n)} \leq C \|f\|_{L^{\frac{n}{\alpha p},\frac{1}{p-1}}(\mathbb{R}^n)}^{\frac{1}{p-1}} \tag{19}$$

for every $f \in L^{\frac{n}{\alpha p},\frac{1}{p-1}}(\mathbb{R}^n)$.

3. Applications to PDE's

The results concerning $\mathbf{V}_{\alpha,p}$ presented in the preceding section have applications to estimates for solutions to nonlinear elliptic equations. The link between nonlinear potentials and bounds for PDE's has been elucidated by T. Kilpelainen and J. Maly [KM] as far as solutions are concerned, and by R. Mingione [Mi] and F. Duzaar and R. Mingione [DM1, DM2] as far as their gradient is concerned. These fundamental contributions provide pointwise estimates for local solutions to nonlinear equations, a prototype of which is the p-Laplace equation

$$-\mathrm{div}(|\nabla u|^{p-2} \nabla u) = f \tag{20}$$

in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, in terms of a Wolff potential of the datum f. Given $p \in (1, \infty)$ and $\alpha \in (0, n)$, the Wolff potential $\mathbf{W}_{\alpha,p} f$ of a measurable function f in Ω is defined as

$$\mathbf{W}_{\alpha,p} f(x) = \int_0^\infty \left(r^{-n+\alpha p} \int_{B(x,r)} |f(y)| dy \right)^{\frac{1}{p-1}} \frac{dr}{r} \qquad \text{for } x \in \mathbb{R}^n. \tag{21}$$

Here, and in what follows, f is continued by 0 outside Ω, and $B(x, r)$ denotes the ball centered at x and with radius r.

[KM, Theorem 1.6] entails that if $1 < p < n$, $f \in W^{-1,p'}(\Omega)$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local weak solution to equation (20), then for any open bounded sets Ω' and Ω'' such that $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ there exists a constant $C = C(p, n, \Omega', \Omega'')$ such that

$$|u(x)| \leq C \bigg(\int_{\Omega'} |u|^{p-1} dx \bigg)^{\frac{1}{p-1}} + C \mathbf{W}_{1,p} f(x) \qquad \text{for a.e. } x \in \Omega''. \tag{22}$$

On the other hand, by [AM, Theorem 3.1], if $\alpha p < n$ there exists a constant $C = C(\alpha, n)$ such that

$$\mathbf{W}_{\alpha,p} f(x) \leq C \mathbf{V}_{\alpha,p} f(x) \qquad \text{for } x \in \mathbb{R}^n. \tag{23}$$

Combining estimates (22) and (23) with Theorem 2.1 yields Theorem 3.1 below, whose content is a local bound for the rearrangement of a solution u to (20) in terms of a norm of u and of the rearrangement of the datum f. Note that rearrangement estimates for solutions to boundary value problems for quasilinear elliptic PDE's are well known, and go back to the pioneering work of V.G. Maz'ya [Ma1, Ma2] and G. Talenti [Ta1, Ta2]. The novelty here is that an analogous result holds even for local solutions.

Theorem 3.1. *Let $1 < p < n$. Let f, u, Ω, Ω' and Ω'' be as above. Then there exist constants $C = C(p, n, \Omega', \Omega'')$ and $k = k(p, n, |\Omega|)$ such that*

$$(u_{|\Omega''})^*(s) \leq C\|u\|_{L^{p-1}(\Omega')} + C\left[\left(s^{\frac{p}{n}-1}\int_0^{ks} f^*(r)dr\right)^{\frac{1}{p-1}} + \int_{ks}^{|\Omega|} r^{\frac{p'}{n}-1} f^*(r)^{\frac{1}{p-1}} dr\right] \quad \text{for } s > 0. \tag{24}$$

Applications of Theorem 2.1 to rearrangement estimates for the gradient of solutions to (20) lead to even more original results. Indeed, the papers quoted above by Maz'ya and by Talenti do not contain gradient bounds in the form of rearrangements. They only deal with estimates for Lebesgue norms of the gradient which are not stronger than L^p. Moreover, these bounds are established for solutions to boundary value problems, and not for local solutions. Rearrangement estimates for gradients are actually available in the literature, and are due to A. Alvino-V. Ferone-G. Trombetti [AFT]. However, the result of that paper is again only of use in view of applications to inequalities for gradient norms which are weaker than L^p; furthermore it holds for solutions to Dirichlet problems.

Theorem 3.2 below provides us with a rearrangement estimate for the gradient of local solutions to (20), at least when $p \geq 2$. This result follows via a key bound contained in [DM2, Theorem 5.2], ensuring that if $p \geq 2$, $f \in W^{-1,p'}(\Omega)$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local weak solution to equation (20), then for any open sets $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ there exists a constant $C = C(p, n, \Omega', \Omega'')$ such that

$$|\nabla u(x)| \leq C\left(\int_{\Omega'} |\nabla u|^{\frac{p}{2}} dx\right)^{\frac{2}{p}} + C\mathbf{W}_{\frac{1}{p},p} f(x) \qquad \text{for a.e. } x \in \Omega''. \tag{25}$$

Theorem 3.2. *Let $p \geq 2$. Let f, u, Ω, Ω' and Ω'' be as above. Then there exist constants $C = C(p, n, \Omega', \Omega'')$ and $k = k(p, n, |\Omega|)$ such that*

$$(|\nabla u|_{|\Omega''})^*(s) \leq C\|\nabla u\|_{L^{\frac{p}{2}}(\Omega')} + C\left[\left(s^{-\frac{1}{n'}}\int_0^{ks} f^*(r)dr\right)^{\frac{1}{p-1}} + \int_{ks}^{|\Omega|} r^{\frac{1}{n(p-1)}-1} f^*(r)^{\frac{1}{p-1}} dr\right] \quad \text{for } s > 0. \tag{26}$$

Theorems 3.1 and 3.2 can be exploited to derive bounds for general rearrangement invariant norms of u and $|\nabla u|$. For instance, as far as the latter is concerned, the following result holds.

Corollary 3.3. *Let p, f, u, Ω, Ω' and Ω'' be as in Theorem 3.2. Let $X(\Omega)$ and $Y(\Omega)$ be rearrangement invariant spaces. If there exists a constant C_1 such that*

$$\left\| s^{-\frac{1}{n'}} \int_0^s \phi(r)dr \right\|_{\overline{Y}(0,|\Omega|)} \leq C_1 \|\phi\|_{\overline{X}(0,|\Omega|)} \tag{27}$$

and

$$\left\| \left(\int_s^{|\Omega|} r^{\frac{1}{n(p-1)}-1} \phi(r)^{\frac{1}{p-1}}\, dr \right)^{p-1} \right\|_{\overline{Y}(0,|\Omega|)} \leq C_1 \|\phi\|_{\overline{X}(0,|\Omega|)} \tag{28}$$

for every nonnegative non-increasing function $\phi \in \overline{X}(0,|\Omega|)$*, then there exists a constant* C_2 *such that*

$$\|\nabla u\|_{Y(\Omega'')} \leq C_2 \|\nabla u\|_{L^{\frac{p}{2}}(\Omega')} + C_2 \|f\|_{X(\Omega)} \tag{29}$$

for every $f \in X(\Omega)$*.*

Gradient bounds in Orlicz and Lorentz spaces for local solutions to equation (20) follow via Theorem 3.2 and Corollary 3.3 in the same way as Theorems 2.3 and 2.4 follow from Theorem 2.1 and Corollary 2.2. Let us just point out here a special case of the relevant bounds, which provides an integrability condition on f ensuring that $|\nabla u|$ is locally in $L^\infty(\Omega)$, namely that u is locally Lipschitz continuous.

Corollary 3.4. *Let* p *and* u *be as in Theorem* 3.2*. If* $f \in L^{n,\frac{1}{p-1}}_{\text{loc}}(\Omega)$*, then* $|\nabla u| \in L^\infty_{\text{loc}}(\Omega)$*.*

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Solvability Conditions for a Discrete Model of Schrödinger's Equation

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Dedicated to Vladimir Maz'ya on the occasion of his 70*th birthday*

Abstract. Let ω be a Borel measure on $\mathbb{R}^n$, and let $\mathcal{Q}$ denote the dyadic cubes in $\mathbb{R}^n$. For a sequence $s = \{s_Q\}_{Q\in\mathcal{Q}}$ of nonnegative scalars, we consider an operator T defined by $Tu(x) = \int_{\mathbb{R}^n} K(x,y)u(y)\,d\omega(y)$ with kernel $K(x,y) = \sum_{Q\in\mathcal{Q}} s_Q\,\omega(Q)^{-1}\chi_Q(x)\chi_Q(y)$. We obtain conditions for the existence of a solution u to the inhomogeneous equation $u = Tu + \alpha$, which serves as a discrete model for an inhomogeneous, time-independent Schrödinger equation on $\mathbb{R}^n$. Define a discrete Carleson norm

$$\|s\|_\omega = \sup_{Q\in\mathcal{Q}:\omega(Q)\neq 0} \omega(Q)^{-1} \sum_{P\in\mathcal{Q}:P\subseteq Q} |s_P|\,\omega(P),$$

and let $A_Q(x) = \sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P(x)$. If $\|s\|_\omega < \frac{1}{12}$, and

$$\int_{\mathbb{R}^n} \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{6\left(A_Q(x)+A_Q(y)\right)}\chi_Q(x)\chi_Q(y)\,|\alpha(y)|\;d\omega(y) < +\infty$$

$d\omega$-a.e., then there exists u satisfying $u = Tu+\alpha$. Other sufficient conditions are derived. In the converse direction, if $\alpha \geq 0$ and the equation $u = Tu+\alpha$ has a solution $u \geq 0$, then $\|s\|_\omega \leq 1$ and

$$\int_{\mathbb{R}^n} \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{\frac{1}{2}\left(A_Q(x)+A_Q(y)\right)}\chi_Q(x)\chi_Q(y)\,|\alpha(y)|\;d\omega(y) < +\infty$$

$d\omega$-a.e. These results are obtained from bilateral estimates for the kernel of the Neumann series $\sum_{j=0}^{\infty} T^j$.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain. We want to find conditions for the existence of a solution u to any of the following problems:

$$\begin{cases} -\triangle u = V \cdot u + \varphi \text{ on } \Omega, \\ u = g \text{ on } \partial\Omega, \end{cases} \tag{1}$$

$$-\triangle u = V \cdot u + \varphi \text{ on } \mathbb{R}^n, \tag{2}$$

where V, φ, and g are given. Equation (2) is a special case of (1) where $\Omega = \mathbb{R}^n$, and u vanishes at infinity. We are especially interested in the existence of *nonnegative* solutions u for general nonnegative V, φ, and g (measurable functions, or possibly measures).

Kalton and Verbitsky [KV] considered the non-linear equation

$$\begin{cases} -\triangle u = V \cdot u^q + \varphi \text{ on } \Omega, \\ u = g \text{ on } \partial\Omega, \end{cases} \tag{3}$$

for $q > 1$, where $u, V, \varphi \geq 0$ in Ω, $g \geq 0$ on $\partial\Omega$, but remarkably their methods don't apply to the linear versions (1), (2).

We apply the Green's function G for $-\triangle$ on Ω to both sides of (1) to obtain that (1) is equivalent to $u = G(V \cdot u) + G(\varphi) + P(g)$, where P denotes the Poisson integral. This formulation is of the form $u = G(V \cdot u) + \alpha$, where α is given. On $\mathbb{R}^n$, $n \geq 3$, the Green's function for $-\triangle$ is the Newtonian potential $I_2 = (-\triangle)^{-1}$, and in the same way (2) is equivalent to

$$u = I_2(V \cdot u) + \alpha, \tag{4}$$

where $\alpha = I_2(\varphi)$. We focus on (4), although a similar approach applies to (1). We set

$$d\omega(y) = |V(y)|\, dy. \tag{5}$$

We let $\mathcal{Q}$ denote the set of all dyadic cubes $Q = 2^{-\nu}([0,1)^n + j)$ $(\nu \in \mathbb{Z}, j \in \mathbb{Z}^n)$ in $\mathbb{R}^n$, and $\mathcal{Q}_\nu = \{Q \in \mathcal{Q} : \ell(Q) = 2^{-\nu}\}$, where $\ell(Q)$ is the side length of the cube Q. Then

$$\begin{aligned} |I_2(V \cdot u)(x)| &= \left| c_n \int_{\mathbb{R}^n} \frac{u(y)V(y)}{|x-y|^{n-2}}\, dy \right| \leq c_n \int_{\mathbb{R}^n} \frac{|u(y)|}{|x-y|^{n-2}}\, d\omega(y) \\ &= \sum_{\nu \in \mathbb{Z}} c_n \int_{\{y \in \mathbb{R}^n : 2^{-\nu} \leq |x-y| \leq 2^{-\nu+1}\}} \frac{|u(y)|}{|x-y|^{n-2}}\, d\omega(y) \\ &= \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_\nu} c_n \int_{\{y \in \mathbb{R}^n : 2^{-\nu} \leq |x-y| \leq 2^{-\nu+1}\}} \frac{|u(y)|}{|x-y|^{n-2}}\, d\omega(y)\ \ \chi_Q(x), \end{aligned}$$

where χ_E denotes the characteristic function of the set E. We follow C. Fefferman [Fef], Hedberg and Wolff [HW], and Chang, Wilson, and Wolff [CWW] in forming a discrete model for this problem. For an appropriate constant $c > 1$, the last expression is bounded from above by

$$\sum_{\nu\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_\nu} c_n \frac{1}{|Q|^{1-\frac{2}{n}}}\int_{cQ}|u(y)|\,d\omega(y)\ \chi_Q(x)$$
$$=\sum_{\nu\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_\nu} c_n \frac{|Q|_\omega}{|Q|^{1-\frac{2}{n}}}\frac{1}{|Q|_\omega}\int_{cQ}|u(y)|\,d\omega(y)\ \chi_Q(x),$$

where $|Q|$ is the Lebesgue measure of Q and $|Q|_\omega = \int_Q d\omega = \omega(Q)$; there is also a matching lower bound with $c = 1$ (see [HW]). We assume that ω is locally finite. We write

$$s_Q = c_n \frac{|Q|_\omega}{|Q|^{1-\frac{2}{n}}}. \tag{6}$$

Note that each s_Q is nonnegative. For our model, we replace $\int_{cQ}$ by $\int_Q$ and define

$$Tu(x) = \sum_{Q\in\mathcal{Q}} s_Q \frac{1}{|Q|_\omega}\int_Q u(y)\,d\omega(y)\ \chi_Q(x). \tag{7}$$

When $|Q|_\omega = 0$ then $s_Q = 0$ and the term in the sum should be interpreted as 0 for this Q. Note that T is determined by ω and the sequence $s = \{s_Q\}_{Q\in\mathcal{Q}}$, and that $Tu(x) \ge 0$ for all x if $u \ge 0$ $d\omega$-a.e. The domain of T is

$$\mathrm{Dom}(T) = \{u : \int_Q |u(y)|\,d\omega(y) < \infty \text{ for all } Q\in\mathcal{Q} \text{ such that } s_Q \neq 0\}. \tag{8}$$

For $\{s_Q\}_{Q\in\mathcal{Q}}$ defined by (6), $\mathrm{Dom}(T) = L^1_{\mathrm{loc}}(d\omega)$, but we will consider more general nonnegative sequences in Sections 2 and 3. Our model problem is to find conditions for the existence of a solution $u \in \mathrm{Dom}(T)$ to the equation

$$u = Tu + \alpha \tag{9}$$

where we assume $\alpha \in \mathrm{Dom}(T)$.

The formal solution of (9) is $u = \sum_{k=0}^\infty T^k\alpha$. Our concern is to determine conditions under which this sum converges in an appropriate sense. If we assume that $\|T\|_{L^2(\omega)\to L^2(\omega)} < 1$ and $\alpha \in L^2(\omega)$, then $\sum_{k=0}^\infty T^k\alpha$ converges in $L^2(\omega)$ and yields a solution to (9). Our goal is to find pointwise conditions for the existence of a solution to (9) for general $\alpha \in \mathrm{Dom}(T)$.

Observe that we can write

$$Tu(x) = \int_{\mathbb{R}^n} K(x,y)u(y)\,d\omega(y) \tag{10}$$

for

$$K(x,y) = \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega}\chi_Q(x)\chi_Q(y). \tag{11}$$

Operators of this form have been studied by Nazarov, Treil, and Volberg [NTV], among others.

The results obtained in this paper led to bilateral global estimates of Green's function for the (fractional) Schrödinger equation that will be presented in the future papers [FV], [FNV]. There are also certain versions of the discrete model applicable to nonlinear equations.

2. Necessary conditions for the discrete model with positive data

In Sections 2 and 3, we assume that we are given a locally finite, nonnegative measure ω on $\mathbb{R}^n, n \geq 3$, a nonnegative sequence $s = \{s_Q\}_{Q\in\mathcal{Q}}$, and $\alpha \in \mathrm{Dom}(T)$. For T defined by (7), we are looking for conditions for the solvability of the equation $u = Tu + \alpha$.

To look for appropriate conditions, we first consider the case where $\alpha(x) \geq 0$ for all $x \in \mathbb{R}^n$, and consider necessary conditions for the existence of a nonnegative solution u to $u = Tu+\alpha$. One necessary condition is immediate. Since $\alpha(x) \geq 0$, we have $Tu(x) \leq u(x)$ for all x. Schur's lemma implies that T is a bounded operator on $L^2(\omega)$ with

$$\|T\|_{L^2(\omega)\to L^2(\omega)} \leq 1. \tag{12}$$

This condition yields a Carleson condition on the sequence s.

Definition 2.1. *Let ω be a nonnegative measure on $\mathbb{R}^n$. A sequence $b = \{b_Q\}_{Q\in\mathcal{Q}}$ of complex numbers belongs to C_ω if*

$$\|b\|_\omega = \sup_{Q\in\mathcal{Q}:|Q|_\omega\neq 0} \frac{1}{|Q|_\omega} \sum_{P\in\mathcal{Q}:P\subseteq Q} |b_P||P|_\omega < \infty, \tag{13}$$

where, if $|Q|_\omega = 0$, we require $b_P = 0$ for all $P \subseteq Q$.

Note that

$$\sup_{Q\in\mathcal{Q}} |b_Q| \leq \|b\|_\omega, \tag{14}$$

just by looking at the term in the sum corresponding to $P = Q$ in (13).

Lemma 2.2. *Suppose $s = \{s_Q\}_{Q\in\mathcal{Q}}$ is a nonnegative sequence and T as defined by (7) is bounded on $L^2(\omega)$. Then $s \in C_\omega$ and*

$$\|s\|_\omega \leq \|T\|_{L^2(\omega)\to L^2(\omega)}.$$

Proof. Note that

$$\begin{aligned} T\chi_Q(x) &= \sum_{P\in\mathcal{Q}} \frac{s_P}{|P|_\omega} \int_P \chi_Q(y)\, d\omega(y)\chi_P(x) \\ &\geq \sum_{P\in\mathcal{Q}:P\subseteq Q} \frac{s_P}{|P|_\omega} \int_P \chi_Q(y)\, d\omega(y)\chi_P(x) = \sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P(x). \end{aligned}$$

Hence

$$\sum_{P\in\mathcal{Q}:P\subseteq Q} s_P|P|_\omega = \int_Q \sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P\,d\omega \le \int_Q T\chi_Q\,d\omega$$
$$\le \|T\chi_Q\|_{L^2(\omega)}\|\chi_Q\|_{L^2(\omega)} \le \|T\|\|\chi_Q\|^2_{L^2(\omega)} = \|T\||Q|_\omega.$$

Also, if $|Q|_\omega = 0$, then of course $|P|_\omega = 0$ for $P \subseteq Q$, and our definition of T guarantees that $s_P = 0$ for all such P. $\square$

When s is as in (6), the condition $s \in C_\omega$ is equivalent to V. Maz'ya's capacity condition in the theory of the Schrödinger operator (see [M]).

Combining Lemma 2.2 with (12), we see that

$$\|s\|_\omega \le 1 \tag{15}$$

is a necessary condition for the existence of a nonnegative solution to $u = Tu + \alpha$. Observe that (15) depends on ω, but not on α.

For any cube $Q \in \mathcal{Q}$, define

$$A_Q(x) = \sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P(x). \tag{16}$$

Lemma 4.5 below shows that ϵA_Q is locally exponentially integrable for sufficiently small ϵ, if $s \in C_\omega$.

To obtain another necessary condition, suppose that u is a nonnegative solution of $u = Tu + \alpha$. Substituting $Tu + \alpha$ for u on the right repeatedly yields

$$u = Tu + \alpha = T(Tu+\alpha) + \alpha = T^2u + T\alpha + \alpha$$
$$= \cdots = T^{m+1}u + \sum_{j=0}^{m} T^j\alpha \ge \sum_{j=0}^{m} T^j\alpha,$$

since $u \ge 0$ implies $T^{m+1}u \ge 0$. Letting $m \to \infty$ on the right side implies that

$$u(x) \ge \sum_{j=0}^{\infty} T^j\alpha(x), \tag{17}$$

for all x. To obtain a necessary condition, we make a lower estimate on the kernel of $\sum_{j=1}^{\infty} T^j$.

Lemma 2.3. *Suppose $s = \{s_Q\}_{Q\in\mathcal{Q}}$ is a nonnegative sequence and T is defined by (7). Let K_j be the kernel of T^j, in the sense that $T^ju(x) = \int_{\mathbb{R}^n} K_j(x,y)u(y)\,d\omega(y)$. Then*

$$\sum_{j=1}^{\infty} K_j(x,y) \ge \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{\frac{1}{2}(A_Q(x)+A_Q(y))}\chi_Q(x)\chi_Q(y), \tag{18}$$

for all $x, y \in \mathbb{R}^n$.

Proof. We begin with the claim that

$$K_j(x,y) \geq \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} \frac{(A_Q(y))^{j-1}}{(j-1)!} \chi_Q(x)\chi_Q(y). \tag{19}$$

We prove (19) by induction on j. The case $j=1$ follows by the definition of T. Now suppose (19) holds for j. Then by the induction hypothesis,

$$\begin{aligned} K_{j+1}(x,y) &= \int_{\mathbb{R}^n} K_j(x,z)K(z,y)\,d\omega(z) \\ &\geq \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} \frac{(A_Q(z))^{j-1}}{(j-1)!} \chi_Q(x)\chi_Q(z) \sum_{P\in\mathcal{Q}} \frac{s_P}{|P|_\omega} \chi_P(z)\chi_P(y)\,d\omega(z). \end{aligned}$$

Since all terms are positive, we obtain a smaller quantity if we replace $\sum_{P\in\mathcal{Q}}$ by $\sum_{P\in\mathcal{Q}:P\subseteq Q}$ in the inner sum. For $P\subseteq Q$, we have $\chi_Q(z)\chi_P(z)=\chi_P(z)$, and we can write $\chi_P(y)=\chi_P(y)\chi_Q(y)$. We obtain

$$\begin{aligned} K_{j+1}(x,y) \geq \frac{1}{(j-1)!} \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} \int_{\mathbb{R}^n} \sum_{P\in\mathcal{Q}:P\subseteq Q} \frac{s_P}{|P|_\omega} \chi_P(z)\chi_P(y) \\ \times (A_Q(z))^{j-1}\,d\omega(z)\chi_Q(x)\chi_Q(y). \end{aligned}$$

Hence the induction step, and therefore (19) will follow once we prove

$$\int_{\mathbb{R}^n} \sum_{P\in\mathcal{Q}:P\subseteq Q} \frac{s_P}{|P|_\omega} \chi_P(z)\chi_P(y)\,(A_Q(z))^{j-1}\,d\omega(z) \geq \frac{1}{j}\,(A_Q(y))^j\,. \tag{20}$$

To prove (20), the left side is

$$\begin{aligned} &\int_{\mathbb{R}^n} \sum_{P\in\mathcal{Q}:P\subseteq Q} \frac{s_P}{|P|_\omega} \chi_P(z)\chi_P(y) \left(\sum_{R\in\mathcal{Q}:R\subseteq Q} s_R\chi_R(z) \right)^{j-1} d\omega(z) \\ &\geq \sum_{P\in\mathcal{Q}:P\subseteq Q} \frac{s_P}{|P|_\omega} \chi_P(y) \int_{\mathbb{R}^n} \chi_P(z) \left(\sum_{R\in\mathcal{Q}:P\subseteq R\subseteq Q} s_R\chi_R(z) \right)^{j-1} d\omega(z). \end{aligned}$$

In this last expression, note that $\chi_R(z)=1$ for $z\in P$ and $P\subseteq R$, so the left side of (20) is bounded below by

$$\sum_{P\in\mathcal{Q}:P\subseteq Q} \frac{s_P}{|P|_\omega} \chi_P(y) \left(\sum_{R\in\mathcal{Q}:P\subseteq R\subseteq Q} s_R \right)^{j-1} \int_{\mathbb{R}^n} \chi_P(z)\,d\omega(z).$$

The terms $|P|_\omega$ and $\int_{\mathbb{R}^n} \chi_P(z)\, d\omega(z)$ cancel, leaving

$$\begin{aligned}
&\sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P(y)\left(\sum_{R\in\mathcal{Q}:P\subseteq R\subseteq Q} s_R\right)^{j-1}\\
&\quad= \sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P(y)\left(\sum_{R\in\mathcal{Q}:P\subseteq R\subseteq Q} s_R\chi_R(y)\right)^{j-1}\\
&\quad\geq \frac{1}{j}\left(\sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P(y)\right)^{j} = \frac{1}{j}\left(A_Q(y)\right)^j,
\end{aligned}$$

by Lemma 4.2 below. This completes the proof of (19).

Summing (19) over j yields

$$\begin{aligned}
\sum_{j=1}^\infty K_j(x,y) &\geq \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega}\sum_{j=1}^\infty \frac{(A_Q(y))^{j-1}}{(j-1)!}\chi_Q(x)\chi_Q(y)\\
&= \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{A_Q(y)}\chi_Q(x)\chi_Q(y).
\end{aligned}$$

However, the kernel K is symmetric, and hence so is K_j for every j, so

$$\sum_{j=1}^\infty K_j(x,y) = \sum_{j=1}^\infty K_j(y,x) \geq \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{A_Q(x)}\chi_Q(x)\chi_Q(y).$$

Averaging yields

$$\sum_{j=1}^\infty K_j(x,y) \geq \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega}\frac{1}{2}\left(e^{A_Q(x)} + e^{A_Q(y)}\right)\chi_Q(x)\chi_Q(y).$$

Using $(a+b)/2 \geq \sqrt{ab}$ gives (18). □

By (17), a nonnegative solution $u \in \mathrm{Dom}(T)$ of $u = Tu + \alpha$, with $\alpha \geq 0$, must satisfy

$$u(x) \geq \sum_{j=0}^\infty T^j\alpha(x) = \alpha(x) + \int_{\mathbb{R}^n}\sum_{j=1}^\infty K_j(x,y)\alpha(y)\, d\omega(y).$$

Since u must be finite $d\omega$-a.e., (18) implies the necessary condition

$$\int_{\mathbb{R}^n}\sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{\frac{1}{2}(A_Q(x)+A_Q(y))}\chi_Q(x)\chi_Q(y)\alpha(y)\, d\omega(y) < \infty \tag{21}$$

$d\omega$-a.e.

3. Sufficient conditions for the discrete model

In this section we show that conditions of the same type as (15) and (21), although with different constants, are sufficient to guarantee the solvability of $u = Tu + \alpha$. Here we are no longer assuming the positivity of α and we are not requiring the solution u to be positive. Our main point is that there is an upper bound for $\sum_{j=1}^{\infty} K_j$, which is of the same form as the lower bound in Lemma 2.3.

Lemma 3.1. *Suppose* $s = \{s_Q\}_{Q\in\mathcal{Q}} \in C_\omega$ *is a nonnegative sequence and* T *is defined by* (7). *Let* K_j *be the kernel of* T^j. *Suppose*

$$\|s\|_\omega \le \frac{1}{12}, \tag{22}$$

and

$$\beta > \frac{6}{1+\sqrt{1-12\|s\|_\omega}}. \tag{23}$$

Define A_Q *by* (16). *Then*

$$\sum_{j=1}^{\infty} K_j(x,y) \le c \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{\beta(A_Q(x)+A_Q(y))}\chi_Q(x)\chi_Q(y), \tag{24}$$

for all $x, y \in \mathbb{R}^n$.

Proof. Let $L(x,y) = \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{\beta(A_Q(x)+A_Q(y))}\chi_Q(x)\chi_Q(y)$. By the monotonicity of (24) in β, and (22), (23), we may assume that

$$\beta\|s\|_\omega < 1, \quad \text{and} \quad \beta(1-\beta\|\mathrm{s}\|_\omega) > 3.$$

For $\gamma = 3/(\beta(1-\beta\|c\|_\omega)) < 1$, we will prove by induction that

$$K_j(x,y) \le \gamma^{j-1} L(x,y), \tag{25}$$

for all x, y and $j \ge 1$, which implies (24). For $j = 1$, (25) is trivial because $e^{\beta(A_Q(x)+A_Q(y))} \ge 1$. Now suppose (25) for $j-1$. By the induction hypothesis,

$$\begin{aligned} K_j(x,y) &= \int_{\mathbb{R}^n} K_{j-1}(x,z)K(z,y)\,d\omega(z) \le \gamma^{j-2}\int_{\mathbb{R}^n} L(x,z)K(z,y)\,d\omega(z) \\ &= \gamma^{j-2}\int_{\mathbb{R}^n}\sum_{Q\in\mathcal{Q}}\frac{s_Q}{|Q|_\omega}e^{\beta(A_Q(x)+A_Q(z))}\chi_Q(x)\chi_Q(z)\sum_{P\in\mathcal{Q}}\frac{s_P}{|P|_\omega}\chi_P(z)\chi_P(y)\,d\omega(z) \\ &= \gamma^{j-2}\left(I + II\right), \end{aligned}$$

where

$$I = \int_{\mathbb{R}^n}\sum_{Q\in\mathcal{Q}}\frac{s_Q}{|Q|_\omega}e^{\beta(A_Q(x)+A_Q(z))}\chi_Q(x)\chi_Q(z)\sum_{P\in\mathcal{Q}:P\subsetneq Q}\frac{s_P}{|P|_\omega}\chi_P(z)\chi_P(y)\,d\omega(z)$$

and

$$II = \int_{\mathbb{R}^n}\sum_{Q\in\mathcal{Q}}\frac{s_Q}{|Q|_\omega}e^{\beta(A_Q(x)+A_Q(z))}\chi_Q(x)\chi_Q(z)\sum_{P\in\mathcal{Q}:Q\subseteq P}\frac{s_P}{|P|_\omega}\chi_P(z)\chi_P(y)\,d\omega(z).$$

Since $\chi_Q(z)\chi_P(z) = \chi_P(z)$ when $P \subseteq Q$, we have

$$I = \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{\beta A_Q(x)} \sum_{P\in\mathcal{Q}:P\subsetneqq Q} \frac{s_P}{|P|_\omega} \int_P e^{\beta A_Q(z)}\, d\omega(z)\chi_P(y)\chi_Q(x).$$

For $z \in P$ with $P \subseteq Q$,

$$\begin{aligned} e^{\beta A_Q(z)} &= e^{\beta \sum_{R\in\mathcal{Q}:R\subseteq Q} s_R\chi_R(z)} = e^{\beta \sum_{R\in\mathcal{Q}:R\subseteq P} s_R\chi_R(z)} e^{\beta \sum_{R\in\mathcal{Q}:P\subsetneqq R\subseteq Q} s_R\chi_R(z)} \\ &= e^{\beta A_P(z)} e^{\beta \sum_{R\in\mathcal{Q}:P\subsetneqq R\subseteq Q} s_R}, \end{aligned}$$

since $\chi_R(z) = 1$ for $z \in P$ and $P \subseteq R$. Hence

$$\begin{aligned} \int_P e^{\beta A_Q(z)}\, d\omega(z) &= e^{\beta \sum_{R\in\mathcal{Q}:P\subsetneqq R\subseteq Q} s_R} \int_P e^{\beta A_P(z)}\, d\omega(z) \\ &\le \frac{|P|_\omega}{1-\beta\|s\|_\omega} e^{\beta \sum_{R\in\mathcal{Q}:P\subsetneqq R\subseteq Q} s_R}, \end{aligned}$$

by Lemma 4.5 below. Hence, substituting this estimate above,

$$I \le \frac{1}{1-\beta\|s\|_\omega} \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{\beta A_Q(x)} \sum_{P\in\mathcal{Q}:P\subsetneqq Q} s_P e^{\beta \sum_{R\in\mathcal{Q}:P\subsetneqq R\subseteq Q} s_R} \chi_P(y)\chi_Q(x).$$

By Lemma 4.3 below, we have

$$\sum_{P\in\mathcal{Q}:P\subsetneqq Q} s_P e^{\beta \sum_{R\in\mathcal{Q}:P\subsetneqq R\subseteq Q} s_R} \chi_P(y) \le \beta^{-1} e^{\beta A_Q(y)} \chi_Q(y).$$

Therefore

$$I \le \frac{1}{\beta(1-\beta\|s\|_\omega)} \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{\beta(A_Q(x)+A_Q(y))} \chi_Q(x)\chi_Q(y) = \frac{1}{\beta(1-\beta\|s\|_\omega)} L(x,y).$$

To estimate II, we write $\sum_Q \sum_{P:Q\subseteq P} = \sum_P \sum_{Q:Q\subseteq P}$ and note that in this case $\chi_Q(z)\chi_P(z) = \chi_Q(z)$ to obtain

$$II = \sum_{P\in\mathcal{Q}} \frac{s_P}{|P|_\omega} \sum_{Q\in\mathcal{Q}:Q\subseteq P} \frac{s_Q}{|Q|_\omega} e^{\beta A_Q(x)} \int_Q e^{\beta A_Q(z)}\, d\omega(z)\chi_Q(x)\chi_Q(y).$$

By Lemma 4.5 below, $\int_Q e^{\beta A_Q(z)}\, d\omega(z) \le (1-\beta\|s\|_\omega)^{-1}|Q|_\omega$. Hence

$$II \le \frac{1}{1-\beta\|s\|_\omega} \sum_{P\in\mathcal{Q}} \frac{s_P}{|P|_\omega} \sum_{Q\in\mathcal{Q}:Q\subseteq P} s_Q e^{\beta A_Q(x)} \chi_Q(x)\chi_Q(y).$$

By Lemma 4.4 below, $\sum_{Q\in\mathcal{Q}:Q\subseteq P} s_Q e^{\beta A_Q(x)}\chi_Q(x) \le \frac{2}{\beta} e^{\beta A_P(x)}\chi_P(x)$. Therefore

$$II \le \frac{2}{\beta(1-\beta\|s\|_\omega)} \sum_{P\in\mathcal{Q}} \frac{s_P}{|P|_\omega} e^{\beta A_P(x)} \chi_P(x)\chi_P(y) \le \frac{2}{\beta(1-\beta\|s\|_\omega)} L(x,y),$$

because $e^{\beta A_Q(y)} \ge 1$. Putting the estimates for I and II together, we have

$$I + II \le \frac{3}{\beta(1-\beta\|s\|_\omega)} L(x,y) = \gamma L(x,y),$$

so

$$K_j(x,y) \le \gamma^{j-2}\gamma L(x,y) = \gamma^{j-1}L(x,y).$$

This completes the inductive step, and hence establishes (25). □

Remark 3.2. *As an example, note that if* $\|s\|_\omega < \frac{1}{12}$ *and* $\beta = 6$ *then* (22) *and* (23) *hold.*

The function v in the following lemma will dominate our solution u. The lemma shows that the assumption that v is finite $d\omega$-a.e. automatically implies the seemingly stronger result that $v \in \mathrm{Dom}(T)$.

Lemma 3.3. *Suppose* $s = \{s_Q\}_{Q\in\mathcal{Q}} \in C_\omega$ *is a nonnegative sequence and* $\beta > 0$. *Suppose* (22) *and* (23) *hold. Define* A_Q *by* (16). *Suppose* $\alpha \in \mathrm{Dom}(T)$. *Let*

$$v(x) = \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{\beta(A_Q(x)+A_Q(y))}\chi_Q(x)\chi_Q(y)|\alpha(y)|\,d\omega(y), \tag{26}$$

for all $x \in \mathbb{R}^n$. *If* $v(x) < \infty$ $d\omega$*-a.e. then* $v \in \mathrm{Dom}(T)$.

Proof. Observe that

$$v(x) = \sum_{Q\in\mathcal{Q}} \frac{s_Q \chi_Q(x)}{|Q|_\omega} e^{\beta A_Q(x)} \int_Q e^{\beta A_Q(y)}\,|\alpha(y)|\,d\omega(y). \tag{27}$$

Let R be a dyadic cube in $\mathbb{R}^n$ with $s_R \neq 0$. Since $s \in C_\omega$, it follows that $|R|_\omega \neq 0$ (see Definition 2.1). Then by Fubini's theorem,

$$\int_R v(x)\,d\omega(x) = \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} \int_{Q\cap R} e^{\beta A_Q(x)}\,d\omega(x) \int_Q e^{\beta A_Q(y)}|\alpha(y)|\,d\omega(y) \le I + II,$$

where $I = \sum_{Q\in\mathcal{Q}:Q\subseteq R}(\cdots)$ and $II = \sum_{Q\in\mathcal{Q}:R\subseteq Q}(\cdots)$. For $Q \subseteq R$,

$$\int_{Q\cap R} e^{\beta A_Q(x)}\,d\omega(x) = \int_Q e^{\beta A_Q(x)}\,d\omega(x) \le c|Q|_\omega,$$

by Lemma 4.5 below, so

$$\begin{aligned} I &\le c \sum_{Q\in\mathcal{Q}:Q\subset R} s_Q \int_Q e^{\beta A_Q(y)}|\alpha(y)|\,d\omega(y) \\ &= c\int_R \sum_{Q\in\mathcal{Q}:Q\subset R} s_Q\chi_Q(y)e^{\beta A_Q(y)}|\alpha(y)|\,d\omega(y). \end{aligned}$$

By Lemma 4.4 below, $\sum_{Q\in\mathcal{Q}:Q\subset R} s_Q\chi_Q(y)e^{\beta A_Q(y)} \le \frac{2}{\beta}e^{\beta A_R(y)}\chi_R(y)$, hence

$$I \le c\int_R e^{\beta A_Q(y)}|\alpha(y)|\,d\omega(y). \tag{28}$$

However, for any $x \in R$, by considering only the term $Q = R$ in (27),

$$v(x) \ge \frac{s_R}{|R|_\omega} e^{\beta A_R(x)} \int_R e^{\beta A_R(y)}\,|\alpha(y)|\,d\omega(y) \ge \frac{s_R}{|R|_\omega}\int_R e^{\beta A_R(y)}\,|\alpha(y)|\,d\omega(y).$$

Hence from (28) we get

$$I \leq c\frac{|R|_\omega}{s_R}v(x), \tag{29}$$

for all $x \in R$.

Next,

$$II = \sum_{Q\in\mathcal{Q}:R\subseteq Q} \frac{s_Q}{|Q|_\omega} \int_R e^{\beta A_Q(x)}\, d\omega(x) \int_Q e^{\beta A_Q(y)}|\alpha(y)|\, d\omega(y).$$

For $R \subseteq Q$, we have

$$A_Q(x) = \sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P(x) = \sum_{P\in\mathcal{Q}:R\subseteq P\subseteq Q, P\neq R} s_P\chi_P(x) + A_R(x),$$

and $\sum_{P\in\mathcal{Q}:R\subseteq P\subseteq Q,P\neq R} s_P\chi_P(x)$ has the constant value $\sum_{P\in\mathcal{Q}:R\subseteq P\subseteq Q,P\neq R} s_P$ on R. Hence

$$II \leq \sum_{Q\in\mathcal{Q}:R\subseteq Q} \frac{s_Q}{|Q|_\omega} e^{\beta\sum_{P\in\mathcal{Q}:R\subseteq P\subseteq Q} s_P} \int_R e^{\beta A_R(x)}\, d\omega(x) \int_Q e^{\beta A_Q(y)}|\alpha(y)|\, d\omega(y).$$

By Lemma 4.5 below, $\int_R e^{\beta A_R(x)}\, d\omega(x) \leq c|R|_\omega$. Hence

$$II \leq c|R|_\omega \sum_{Q\in\mathcal{Q}:R\subseteq Q} \frac{s_Q}{|Q|_\omega} e^{\beta\sum_{P\in\mathcal{Q}:R\subseteq P\subseteq Q} s_P} \int_Q e^{\beta A_Q(y)}|\alpha(y)|\, d\omega(y). \tag{30}$$

For any $x \in R$, we have from (27) that

$$v(x) \geq \sum_{Q\in\mathcal{Q}:R\subseteq Q} \frac{s_Q}{|Q|_\omega} e^{\beta\sum_{P\in\mathcal{Q}:R\subseteq P\subseteq Q} s_P} \int_Q e^{\beta A_Q(y)}\, |\alpha(y)|\, d\omega(y),$$

because $\chi_Q(x) = 1$ and $A_Q(x) \geq \sum_{P\in\mathcal{Q}:R\subseteq P\subseteq Q} s_P$. Comparing with (30) gives $II \leq c|R|_\omega v(x)$, for all $x \in R$. Combining this with (29) gives

$$\int_R v\, d\omega \leq c|R|_\omega(1 + 1/s_R)v(x), \tag{31}$$

for all $x \in R$. Since $|R|_\omega \neq 0$, and our assumption is that $v(x) < \infty$ $d\omega$-a.e., we can select $x \in R$ such that $v(x) < \infty$. We conclude that $v \in \mathrm{Dom}(T)$. □

Lemma 3.1 easily gives sufficient criteria for the solvability of $u = Tu + \alpha$.

Theorem 3.4. *Let $s = \{s_Q\}_{Q\in\mathcal{Q}}$ be a nonnegative sequence, and let ω be a locally finite, nonnegative weight. Define T by* (7) *and let $\alpha \in \mathrm{Dom}(T)$. Define A_Q for $Q \in \mathcal{Q}$ by* (16)*. Suppose* (22), (23) *hold, and*

$$\int_{\mathbb{R}^n} \sum_{Q\in\mathcal{Q}} \frac{s_Q}{|Q|_\omega} e^{\beta(A_Q(x)+A_Q(y))}\chi_Q(x)\chi_Q(y)\, |\alpha(y)|\, d\omega(y) < +\infty \tag{32}$$

$d\omega$-a.e. Then there exists $u \in \mathrm{Dom}(T)$ satisfying $u = Tu + \alpha$.

Proof. By assumption, v, defined by (26), is finite $d\omega$-a.e. By Lemma 3.1,

$$\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} K_j(x,y)\,|\alpha(y)|\; d\omega(y) \le v(x) < \infty \quad d\omega - a.e.$$

In other words, $\sum_{j=0}^{\infty} T^j \alpha(x) = \alpha(x) + \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} K_j(x,y)\alpha(y)\, d\omega(y)$ is absolutely convergent $d\omega$-a.e. Then $u = \sum_{j=0}^{\infty} T^j \alpha$ is defined and $|u| \le |\alpha| + v$. By Lemma 3.3, $v \in \mathrm{Dom}(T)$, and $\alpha \in \mathrm{Dom}(T)$ by assumption. Hence $u \in \mathrm{Dom}(T)$. Moreover, u solves the equation $u = Tu + \alpha$. □

We remark on some relatively simple conditions on the measure ω, the sequence $\{s_Q\}_{Q\in\mathcal{Q}}$, and the function α, which guarantee the finiteness $d\omega$-a.e. of the function v in Theorem 3.4.

Corollary 3.5. *Let $s = \{s_Q\}_{Q\in\mathcal{Q}}$ be a nonnegative sequence, and let ω be a nonnegative weight. Define T by* (7) *and let $\alpha \in \mathrm{Dom}(T)$. Define A_Q for $Q \in \mathcal{Q}$ by* (16). *Suppose* (22), (23) *hold,*

$$\sup_{Q\in\mathcal{Q}:|Q|_\omega\neq 0} \frac{1}{|Q|_\omega} \int_Q e^{\beta A_Q(y)} |\alpha(y)|\, d\omega(y) < \infty, \tag{33}$$

and

$$\sum_{Q\in\mathcal{Q}:R\subseteq Q} s_Q < \infty \text{ for all } R \in \mathcal{Q} \text{ such that } |R| = 1. \tag{34}$$

Then (32) *holds, and hence there exists $u \in \mathrm{Dom}(T)$ satisfying $u = Tu + \alpha$.*

Proof. By (27) and (33),

$$v(x) \le c \sum_{Q\in\mathcal{Q}} s_Q \chi_Q(x) e^{\beta A_Q(x)}.$$

Letting $|Q| \to \infty$ in Lemma 4.4 below shows that

$$\sum_{Q\in\mathcal{Q}} s_Q \chi_Q(x) e^{\beta A_Q(x)} \le c e^{\beta \sum_{Q\in\mathcal{Q}} s_Q \chi_Q(x)}.$$

Hence the finiteness of v $d\omega$-a.e. follows if we show $\sum_{Q\in\mathcal{Q}} s_Q \chi_Q < \infty$ $d\omega$-a.e. Let R be the dyadic cube of side length 1 which contains x. Then $\sum_{Q\in\mathcal{Q}:R\subseteq Q} s_Q < \infty$ by (34). Also, however,

$$\int_R \sum_{P\in\mathcal{Q}:P\subseteq R} s_P \chi_P \, d\omega = \sum_{P\in\mathcal{Q}:P\subseteq R} s_P |P|_\omega \le \|s\|_\omega |R|_\omega < \infty,$$

hence $\sum_{P\in\mathcal{Q}:P\subseteq R} s_P \chi_P(x) < \infty$ for ω-a.e. $x \in R$. □

Remark 3.6. *We note that if $\alpha \in L^\infty(d\omega)$, then* (33) *holds automatically, by Lemma* 4.5 *in the next section. Also, if $s_Q = |Q|_\omega/|Q|^{1-2/n}$, as in Section* 1, *and if $|Q|_\omega \le c_j$ when the length of Q is 2^j, then* (34) *holds if $\sum_{j=0}^{\infty} 2^{-j(n-2)} c_j < \infty$. This estimate is automatic if ω is a finite measure.*

4. Summation by parts lemmas

In the following four lemmas, a nonnegative sequence $s = \{s_P\}_{P\in\mathcal{Q}}$ and a cube $Q \in \mathcal{Q}$ are given, and, for $\nu = 0, 1, 2, \ldots$, we use the notation

$$f_\nu(x) = \sum_{P\in\mathcal{Q}:P\subseteq Q,\ell(P)=2^{-\nu}\ell(Q)} s_P\chi_P(x). \tag{35}$$

Lemma 4.1. *Let $s = \{s_P\}_{P\in\mathcal{Q}}$ be a nonnegative sequence and let $Q \in \mathcal{Q}$. Then*

$$\left(\sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P\right)^m \le m \sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P \left(\sum_{P'\in\mathcal{Q}:P'\subseteq P} s_{P'}\chi_{P'}\right)^{m-1}, \tag{36}$$

for $m = 1, 2, \ldots$.

Proof. The left-hand side of (36) is $\left(\sum_{\nu=0}^\infty f_\nu\right)^m$. We fix a positive integer M temporarily. Then

$$\left(\sum_{\nu=0}^M f_\nu\right)^m = \sum_{\nu=0}^M \left[\left(\sum_{k=\nu}^M f_k\right)^m - \left(\sum_{k=\nu+1}^M f_k\right)^m\right].$$

For $B < A$, we apply the inequality

$$A^m - B^m = (A-B)\sum_{k=0}^{m-1} A^k B^{m-1-k} \le (A-B)\sum_{k=0}^{m-1} A^{m-1} = mA^{m-1}(A-B), \tag{37}$$

to obtain

$$\left(\sum_{\nu=0}^M f_\nu\right)^m \le m\sum_{\nu=0}^M f_\nu \left(\sum_{k=\nu}^M f_k\right)^{m-1} \le m\sum_{\nu=0}^\infty f_\nu \left(\sum_{k=\nu}^\infty f_k\right)^{m-1},$$

and the last expression is the right side of (36). Letting $M \to \infty$ on the left side completes the proof. □

Lemma 4.2. *Let $s = \{s_P\}_{P\in\mathcal{Q}}$ be a nonnegative sequence and let $Q \in \mathcal{Q}$. Then*

$$\left(\sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P\right)^m \le m \sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P \left(\sum_{R\in\mathcal{Q}:P\subseteq R\subseteq Q} s_R\chi_R\right)^{m-1}, \tag{38}$$

for $m = 1, 2, \ldots$.

Proof. The left-hand side of (38) is $\left(\sum_{\nu=0}^\infty f_\nu\right)^m$. We fix a positive integer M temporarily. Then using (37) again,

$$\left(\sum_{\nu=0}^M f_\nu\right)^m = \sum_{\nu=0}^M \left[\left(\sum_{k=0}^{\nu} f_k\right)^{m-1} - \left(\sum_{k=0}^{\nu-1} f_k\right)^{m-1}\right]$$
$$\le m\sum_{\nu=0}^M f_\nu \left(\sum_{k=0}^{\nu} f_k\right)^{m-1} \le m\sum_{\nu=0}^\infty f_\nu \left(\sum_{k=0}^{\nu} f_k\right)^{m-1},$$

and the last expression is the right side of (38). Letting $M \to \infty$ on the left side completes the proof. □

Lemma 4.3. *Suppose $s = \{s_P\}_{P\in\mathcal{Q}}$ is a nonnegative sequence, $Q \in \mathcal{Q}$, and $\beta > 0$. Then*

$$\sum_{P\in\mathcal{Q}:P\subseteq Q, P\neq Q} s_P e^{\beta \sum_{R\in\mathcal{Q}:P\subseteq R\subseteq Q, R\neq P} s_R} \chi_P \leq \frac{1}{\beta} e^{\beta A_Q} \chi_Q. \tag{39}$$

Proof. The left-hand side of (39) is $\sum_{\nu=1}^{\infty} f_\nu e^{\beta \sum_{k=0}^{\nu-1} f_k}$. Fix a positive integer M. Applying the inequality $t \leq e^t - 1$ for $t > 0$ with $t = \beta f_\nu$, we have

$$\begin{aligned}\sum_{\nu=1}^{M} f_\nu e^{\beta \sum_{k=0}^{\nu-1} f_k} &\leq \frac{1}{\beta} \sum_{\nu=1}^{M} \left(e^{\beta f_\nu} - 1\right) e^{\beta \sum_{k=0}^{\nu-1} f_k} = \frac{1}{\beta} \sum_{\nu=1}^{M} \left(e^{\beta \sum_{k=0}^{\nu} f_k} - e^{\beta \sum_{k=0}^{\nu-1} f_k}\right) \\ &= \frac{1}{\beta} \left(e^{\beta \sum_{k=0}^{M} f_k} - e^{\beta f_0}\right) \leq \frac{1}{\beta} e^{\beta \sum_{k=0}^{M} f_k} \leq \frac{1}{\beta} e^{\beta \sum_{k=0}^{\infty} f_k}.\end{aligned}$$

The last expression is the right-hand side of (39), so letting $M \to \infty$ yields (39). □

Lemma 4.4. *Suppose a nonnegative sequence $s = \{s_P\}_{P\in\mathcal{Q}} \in C_\omega$ and $\beta > 0$ satisfy $\beta \|s\|_\omega < 1$. Suppose $Q \in \mathcal{Q}$. Then*

$$\sum_{P\in\mathcal{Q}:P\subseteq Q} s_P \chi_P e^{\beta A_P} \leq \frac{2}{\beta} e^{\beta A_Q} \chi_Q. \tag{40}$$

Proof. The left-hand side of (40) is $\sum_{\nu=0}^{\infty} f_\nu e^{\beta \sum_{k=\nu}^{\infty} f_k}$. We fix a positive integer M temporarily. Note that $|f_\nu(x)| \leq \sup_{Q\in\mathcal{Q}} s_Q \leq \|s\|_\omega$ by (14), hence $0 \leq \beta f_\nu(x) \leq 1$ for all x. Applying the inequality $te^t \leq 2(e^t - 1)$ for $0 \leq t \leq 1$ with $t = \beta f_\nu(x)$, we obtain

$$\begin{aligned}\sum_{\nu=0}^{M} f_\nu(x) e^{\beta \sum_{k=\nu}^{M} f_k(x)} &= \frac{1}{\beta} \sum_{\nu=0}^{M} \beta f_\nu(x) e^{\beta f_\nu(x)} e^{\beta \sum_{k=\nu+1}^{M} f_k(x)} \\ &\leq \frac{2}{\beta} \sum_{\nu=0}^{M} \left(e^{\beta f_\nu(x)} - 1\right) e^{\beta \sum_{k=\nu+1}^{M} f_k(x)} = \frac{2}{\beta} \sum_{\nu=0}^{M} \left(e^{\beta \sum_{k=\nu}^{M} f_k(x)} - e^{\beta \sum_{k=\nu+1}^{M} f_k(x)}\right) \\ &= \frac{2}{\beta} \left(e^{\beta \sum_{k=0}^{M} f_k(x)} - 1\right) \leq \frac{2}{\beta} e^{\beta \sum_{k=0}^{M} f_k(x)} \leq \frac{2}{\beta} e^{\beta \sum_{k=0}^{\infty} f_k(x)}.\end{aligned}$$

Note that $e^{\beta \sum_{k=0}^{\infty} f_k}$ is the right-hand side of (40). Taking the limit as $M \to \infty$ on the left of the inequality above gives (40). □

Lemma 4.5. *Let $s = \{s_Q\}_{Q\in\mathcal{Q}} \in C_\omega$, where s is a nonnegative sequence. Then*

$$\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|_\omega} \int_Q (A_Q)^m \, d\omega \leq m! \|s\|_\omega^m, \tag{41}$$

for all positive integers m. Also, if $0 < \epsilon < 1/\|s\|_\omega$, *then*

$$\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|_\omega} \int_Q e^{\epsilon A_Q}\, d\omega \le \frac{1}{1-\epsilon\|s\|_\omega}. \tag{42}$$

Proof. Fix $Q \in \mathcal{Q}$. Iterating the estimate (36) gives

$$(A_Q)^m \le m! \sum_{P_1\in\mathcal{Q}:P_1\subseteq Q} s_{P_1}\chi_{P_1} \cdots \sum_{P_m\in\mathcal{Q}:P_{m-1}\subseteq P_n} s_{P_n}\chi_{P_n}.$$

For $P_n \subseteq P_{n-1} \subseteq \cdots \subseteq P_1$, we have $\chi_{P_1}\chi_{P_2}\cdots\chi_{P_n} = \chi_{P_n}$, hence integrating the estimate above over Q gives

$$\int_Q (A_Q)^m \, d\omega$$

$$\le m! \sum_{P_1\in\mathcal{Q}:P_1\subseteq Q} s_{P_1} \cdots \sum_{P_{m-1}\in\mathcal{Q}:P_{m-1}\subseteq P_{m-2}} s_{P_{m-1}} \cdot \sum_{P_m\in\mathcal{Q}:P_m\subseteq P_{m-1}} s_{P_m}|P_m|_\omega.$$

Substituting the estimate $\sum_{P_m\in\mathcal{Q}:P_m\subseteq P_{m-1}} s_{P_m}|P_m|_\omega \le \|s\|_\omega |P_{m-1}|_\omega$ and iterating yields

$$\int_Q (A_Q)^m \, d\omega \le m!\|s\|_\omega^{m-1} \sum_{P_1\in\mathcal{Q}:P_1\subseteq Q} s_{P_1}|P_1|_\omega \le m!\|s\|_\omega^m |Q|_\omega,$$

which gives (41).

Expanding e^t in a power series, interchanging summation and integration, and applying (41) gives

$$\int_Q e^{\epsilon A_Q}\, d\omega = \sum_{n=0}^\infty \frac{\epsilon^n}{n!} \int_Q \left(\sum_{P\in\mathcal{Q}:P\subseteq Q} s_P\chi_P \right)^n d\omega$$

$$\le \sum_{m=0}^\infty \frac{\epsilon^m}{m!} m!\|s\|_\omega^m |Q|_\omega = \frac{1}{1-\epsilon\|s\|_\omega}|Q|_\omega. \qquad \square$$

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An Algebra of Shift-invariant Singular Integral Operators with Slowly Oscillating Data and Its Application to Operators with a Carleman Shift

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To Professor V.G. Maz'ya on the occasion of his 70*th birthday*

Abstract. The paper is devoted to studying Banach algebras of shift-invariant singular integral operators with slowly oscillating coefficients and their extensions by shift operators associated with iterations of a slowly oscillating Carleman shift generating a finite cyclic group. Both algebras are contained in the Banach algebra of bounded linear operators on a weighted Lebesgue space with a slowly oscillating Muckenhoupt weight over a composed slowly oscillating Carleson curve. By applying the theory of Mellin pseudodifferential operators, Fredholm symbol calculi for these algebras and Fredholm criteria and index formulas for their elements are established in terms of their Fredholm symbols.

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1. Introduction

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space X, let $\mathcal{K}(X)$ be the closed two-sided ideal of all compact operators in $\mathcal{B}(X)$, and let $\mathcal{B}^\pi(X) = \mathcal{B}(X)/\mathcal{K}(X)$ be the Calkin algebra of the cosets $A^\pi = A + \mathcal{K}$ where $A \in \mathcal{B}(X)$. An operator $A \in \mathcal{B}(X)$ is said to be *Fredholm*, if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional. In that case the number $\operatorname{Ind} A = \dim\ker A - \dim\ker A^*$ is referred to as the *index* of A (see [4]).

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Let Γ be an oriented rectifiable curve in the complex plane, and let $L^p(\Gamma, w)$ be the weighted Lebesgue space with the norm

$$\|f\|_{L^p(\Gamma,w)} := \left(\int_\Gamma |f(\tau)|^p w(\tau)^p |d\tau| \right)^{1/p}$$

where $1 < p < \infty$ and $w : \Gamma \to [0, \infty]$ is a measurable function such that $w^{-1}(\{0, \infty\})$ has measure zero. As is known (see, e.g., [8], [5], [6] and [1]), the Cauchy singular integral operator S_Γ, given for $f \in L^1(\Gamma)$ and almost all $t \in \Gamma$ by

$$(S_\Gamma f)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma(t,\varepsilon)} \frac{f(\tau)}{\tau - t} \, d\tau, \quad \Gamma(t, \varepsilon) := \big\{ \tau \in \Gamma : |\tau - t| < \varepsilon \big\}, \tag{1.1}$$

is bounded on the space $L^p(\Gamma, w)$ if and only if $p \in (1, \infty)$ and $w \in A_p(\Gamma)$, that is,

$$\sup_{t \in \Gamma} \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \left(\int_{\Gamma(t,\varepsilon)} w(\tau)^p |d\tau| \right)^{1/p} \left(\int_{\Gamma(t,\varepsilon)} w(\tau)^{-q} |d\tau| \right)^{1/q} < \infty, \tag{1.2}$$

where $1/p + 1/q = 1$. If the Muckenhoupt condition (1.2) holds, then Hölder's inequality implies that Γ is a Carleson (Ahlfors-David) curve, that is (see [1]),

$$\sup_{t \in \Gamma} \sup_{\varepsilon > 0} |\Gamma(t, \varepsilon)| / \varepsilon < \infty, \tag{1.3}$$

where $|\Gamma(t, \varepsilon)|$ stands for the Lebesgue (length) measure of $\Gamma(t, \varepsilon)$.

Let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N$ be a star-like Carleson curve consisting of unbounded oriented simple open arcs

$$\Gamma_k := \big\{ \tau = t + re^{i\theta_k(r)} : r \in \mathbb{R}_+ \big\} \tag{1.4}$$

and the common nodes t and ∞ where θ_k are real-valued functions, let w be a Muckenhoupt weight in $A_p(\Gamma)$ where $1 < p < \infty$, and let $\mathcal{A}_\Gamma$ be the set of all orientation-preserving diffeomorphisms α of each arc Γ_k ($k = 1, 2, \ldots, N$) onto itself and such that

$$\ln |\alpha'| \in C_b(\Gamma), \quad (w \circ \alpha)/w, \ w/(w \circ \alpha) \in L^\infty(\Gamma). \tag{1.5}$$

Then the shift operator V_α given by $V_\alpha f = f \circ \alpha$ and its inverse operator V_α^{-1} are bounded on the space $L^p(\Gamma, w)$. Let $\mathfrak{S} := \mathfrak{S}_{p,\Gamma,w}$ denote the Banach subalgebra of $\mathcal{B}(L^p(\Gamma, w))$ generated by all multiplication operators cI with $c \in SO(\Gamma)$ and by all generalized singular integral operators $V_\alpha S_\Gamma V_\alpha^{-1}$ with $\alpha \in \mathcal{A}_\Gamma$.

The present paper is devoted to studying the Banach algebra $\mathfrak{S}$ consisting of shift-invariant singular integral operators with slowly oscillating data described in Section 2 and the Banach algebra $\mathfrak{B} := \mathfrak{B}_{p,\Gamma,w,\gamma}$ generated by the operators $C \in \mathfrak{S}$ and by the shift operator $V_\gamma \in \mathcal{B}(L^p(\Gamma, w))$ where γ is a Carleman shift on Γ, that is [17], γ generates a finite cyclic group. The algebra $\mathfrak{S}$ arises in investigations of nonlocal singular integral operators with slowly oscillating coefficients and discrete groups of shifts having slowly oscillating derivatives on weighted Lebesgue spaces with slowly oscillating Muckenhoupt weights over slowly oscillating composed Carleson curves. To study such operators, as well as operators $B \in \mathfrak{B}$, we apply the theory of Mellin pseudodifferential operators.

The C^*-algebra of singular integral operators with slowly oscillating data on the space $L^2(\Gamma, w)$ was studied in [3] (also see [2]). A non-closed algebra of generalized singular integral operators with slowly oscillating data was studied in [12] by using Mellin pseudodifferential operators with compound slowly oscillating $V_0(\mathbb{R})$-valued symbols, where $V_0(\mathbb{R})$ is the Banach algebra of all absolutely continuous functions of bounded total variation. For the classic theory of singular integral operators with a Carleman shift see [17], [9] and the references therein. Some applications of Mellin pseudodifferential operators to singular integral operators with shifts were done in [13]–[15].

The paper is organized as follows. In Section 2 we describe slowly oscillating data. In Section 3 we collect necessary results of [11]–[12] on pseudodifferential operators with slowly oscillating $V(\mathbb{R})$-valued symbols where $V(\mathbb{R})$ is the Banach algebra of continuous functions of bounded total variation on $\mathbb{R}$. In Section 4 we construct a Banach algebra $\widehat{\mathfrak{A}}_p$ of Fredholm symbols which is isomorphic to the Banach algebra $\mathfrak{A}_p$ of Mellin pseudodifferential operators with slowly oscillating $V(\mathbb{R})$-valued symbols on the space $L^p(\mathbb{R}_+, r^{-1}dr)$, and adapt a Fredholm criterion and an index formula obtained in [11] (cf. also [18], [19]) in terms of Fredholm symbols to the operators $A \in \mathfrak{A}_p$. In Section 5 we apply the results on Mellin pseudodifferential operators to shift-invariant (generalized) singular integral operators. In Section 6 we construct an algebra $\widehat{\mathfrak{S}}$ of Fredholm symbols isomorphic to the Banach algebra $\mathfrak{S}$ of generalized singular integral operators and obtain a Fredholm criterion and an index formula for the operators $B \in \mathfrak{S}$. Finally, in Section 7 we construct a Fredholm symbol calculus for the algebra $\mathfrak{B}$ and obtain a Fredholm criterion and an index formula for generalized singular integral operators with a Carleman shift γ on the space $L^p(\Gamma, w)$ in the case of slowly oscillating data.

2. Slowly oscillating data

Following [3], [14] and [15], we introduce the slowly oscillating data.

Slowly oscillating functions. Let $\mathbb{R}_+ = (0, \infty)$ and let $SO(\mathbb{R}_+)$ stand for the set of all functions $a \in C_b(\mathbb{R}_+) := C(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ which are slowly oscillating at 0 and ∞, that is (see, e.g., [20]), satisfy the condition

$$\lim_{r \to s} \max \Big\{ |a(x) - a(y)| : x, y \in [r, 2r] \Big\} = 0, \quad \text{for } s \in \{0, \infty\}. \tag{2.1}$$

It is clear that (2.1) is equivalent to the condition

$$\lim_{r \to s} \max \Big\{ |a(r) - a(\nu r)| : \nu \in [\lambda^{-1}, \lambda] \Big\} = 0, \quad \text{for } s \in \{0, \infty\},$$

with any $\lambda > 1$. Obviously, $SO(\mathbb{R}_+)$ is a unital C^*-subalgebra of $L^\infty(\mathbb{R}_+)$.

Let $M(\mathcal{A})$ be the maximal ideal space of a unital commutative C^*-algebra $\mathcal{A}$. Identifying the points $t \in \overline{\mathbb{R}}_+ := [0, +\infty]$ with the evaluation functionals $t(f) = f(t)$ for $f \in C(\overline{\mathbb{R}}_+)$, we get $M(C(\overline{\mathbb{R}}_+)) = \overline{\mathbb{R}}_+$. Consider the fibers

$$M_s(SO(\mathbb{R}_+)) := \big\{ \xi \in M(SO(\mathbb{R}_+)) : \xi\big|_{C(\overline{\mathbb{R}}_+)} = s \big\}$$

of the maximal ideal space $M(SO(\mathbb{R}_+))$ over the points $s \in \{0,\infty\}$. These fibers are characterized by the following two assertions obtained similarly to [3, Section 2] and [11, Propositions 2.4 and 2.5]).

Proposition 2.1. [14, Propositions 2.1] *We have*

$$\mathfrak{M} := M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+)) = \operatorname{clos}_{SO^*}\mathbb{R}_+ \setminus \mathbb{R}_+ \tag{2.2}$$

where $\operatorname{clos}_{SO^*}\mathbb{R}_+$ *is the weak-star closure of* $\mathbb{R}_+$ *in the dual space of* $SO(\mathbb{R}_+)$.

Proposition 2.2. [14, Proposition 2.2] *Let* $\{a_k\}_{k=1}^\infty$ *be a countable subset of* $SO(\mathbb{R}_+)$ *and* $s \in \{0,\infty\}$. *For each* $\xi \in M_s(SO(\mathbb{R}_+))$ *there is a sequence* $\{x_n\} \subset \mathbb{R}_+$ *such that* $x_n \to s$ *as* $n \to \infty$ *and*

$$\xi(a_k) = \lim_{n\to\infty} a_k(x_n) \quad \textit{for all} \;\; k = 1,2,\dots \tag{2.3}$$

Conversely, if $\{x_n\} \subset \mathbb{R}_+$, $x_n \to s$ *as* $n \to \infty$, *and the limits* $\lim_{n\to\infty} a_k(x_n)$ *exist for all* $k = 1,2,\dots$, *then there is a* $\xi \in M_s(SO(\mathbb{R}_+))$ *such that* (2.3) *holds.*

In what follows we write $a(\xi) := \xi(a)$ for every $a \in SO(\mathbb{R}_+)$ and every $\xi \in \mathfrak{M}$.

Slowly oscillating star-like curves. Suppose $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_N$ where Γ_k are unbounded oriented simple open arcs given by (1.4), and there are numbers

$$0 < m_1 < M_1 < m_2 < M_2 < \dots < m_{N-1} < M_{N-1} < 2\pi$$

such that $m_k < \theta_{k+1}(r) - \theta_1(r) < M_k$ for all $k = 1,2,\dots,N-1$, whence the closures $\overline{\Gamma}_k$ of Γ_k have the only common points t and ∞. The star-like curve Γ is called slowly oscillating if the real-valued functions $\theta_1,\dots,\theta_N$ are in $C^3(\mathbb{R}_+)$ and

$$(rD_r)^j\theta_1,\,\dots,\,(rD_r)^j\theta_N \in SO(\mathbb{R}_+) \quad \text{for all} \;\; j = 1,2,3, \tag{2.4}$$

where $(rD_r)\theta = r\theta'(r)$. Hence $(rD_r)^j\theta_1,\,\dots,\,(rD_r)^j\theta_N \in C_b(\mathbb{R}_+)$ for $j = 1,2,3$, and from [10, Proposition 3.3] it follows that for $s \in \{0,\infty\}$,

$$\lim_{r\to s}\big[(rD_r)^j\theta_k\big](r) = 0 \quad \text{for} \;\; j = 2,3 \;\; \text{and all} \;\; k = 1,2,\dots,N.$$

Setting $\theta_{N+1} := \theta_1$, we also assume that

$$\lim_{r\to s}\big(r\theta'_{k+1}(r) - r\theta'_k(r)\big) = 0 \quad \text{for} \;\; s \in \{0,\infty\} \;\; \text{and} \;\; k = 1,2,\dots,N. \tag{2.5}$$

Note that the functions θ_k may be unbounded, while the boundedness of the functions $\theta_{k+1}-\theta_k$ together with the requirement that $r\big(\theta'_{k+1}(r)-\theta'_k(r)\big)$ be slowly oscillating at the points $s \in \{0,\infty\}$ implies that the functions $\theta_{k+1}-\theta_k$ themselves are slowly oscillating on $\mathbb{R}_+$ if and only if (2.5) holds (see [10, Propositions 3.2 and 3.3]). Obviously, by (2.4), Γ satisfies (1.3) and hence is a Carleson curve (see [3]).

Slowly oscillating weights. Let Γ be a slowly oscillating star-like curve given by (1.4). We call a function $w : \Gamma \to (0,\infty)$ a slowly oscillating weight (at t and ∞) if

$$w(\tau) = e^{v(|\tau-t|)} \quad \text{for} \;\; \tau \in \Gamma, \tag{2.6}$$

where v is a real-valued function in $C^3(\mathbb{R}_+)$ and the functions $(rD_r)^jv$ are in $SO(\mathbb{R}_+)$ for all $j = 1,2,3$. Given $p \in (1,\infty)$, one can show (see, e.g., [1, Theorem 2.36] and [16, Section 5]) that $w \in A_p(\Gamma)$ if and only if

$$-1/p < \liminf_{r\to s} rv'(r) \le \limsup_{r\to s} rv'(r) < 1/q \quad \text{for } s \in \{0,\infty\}.$$

We denote by A_p^{SO} the set of all pairs (Γ, w) such that Γ is a slowly oscillating star-like curve and w is a slowly oscillating weight in $A_p(\Gamma)$. Thus, if $1 < p < \infty$ and $(\Gamma, w) \in A_p^{SO}$, then the Cauchy singular integral operator S_Γ is bounded on the space $L^p(\Gamma, w)$.

Slowly oscillating shifts. Let $(\Gamma, w) \in A_p^{SO}$ and let α be an orientation-preserving diffeomorphism of each arc Γ_k $(k = 1, 2, \dots, N)$ onto itself that satisfies (1.5) and hence implies the boundedness of the operators V_α and V_α^{-1} on the space $L^p(\Gamma, w)$. We call α a slowly oscillating shift (at t and ∞) if for every $k = 1, 2, \dots, N$,

$$\alpha\big(t + re^{i\theta_k(r)}\big) = t + re^{\omega_k(r)} \exp\big(i\theta_k\big(re^{\omega_k(r)}\big)\big) \quad \text{for all } r \in \mathbb{R}_+, \tag{2.7}$$

where ω_k are real-valued functions in $C^3(\mathbb{R}_+)$ and the functions $(rD_r)^j\omega_k$ belong to $SO(\mathbb{R}_+)$ for all $j = 0, 1, 2, 3$. Observe that the slow oscillation of ω_k and $(rD_r)\omega_k$ at t and ∞ is equivalent to the property:

$$\lim_{r\to 0} r\omega_k'(r) = 0, \quad \lim_{r\to\infty} r\omega_k'(r) = 0 \quad (k = 1, 2, \dots, N). \tag{2.8}$$

If α is a slowly oscillating shift then, according to the relations

$$\alpha'(\tau) = \frac{1 + r\omega_k'(r)}{1 + ir\theta_k'(r)}\Big(1 + ire^{\omega_k(r)}\theta_k'\big(re^{\omega_k(r)}\big)\Big) \exp\Big(\omega_k(r) + i\theta_k\big(re^{\omega_k(r)}\big) - i\theta_k(r)\Big)$$

for $\tau = t + re^{i\theta_k(r)} \in \Gamma_k$, every function $r \mapsto \alpha'\big(t + re^{i\theta_k(r)}\big)$ for $k = 1, 2, \dots, N$ belongs to $SO(\mathbb{R}_+)$. Since $\ln|\alpha'| \in C_b(\Gamma)$, we conclude due to (2.8) that

$$\inf_{r\in\mathbb{R}_+} \big(1 + r\omega_k'(r)\big) > 0 \quad \text{for all } k = 1, 2, \dots, N.$$

Let $\mathcal{A}_\Gamma$ be the set of all slowly oscillating shifts $\alpha : \Gamma \to \Gamma$ described above and satisfying (1.5) and (2.7)–(2.8). Obviously, the identity shift belongs to $\mathcal{A}_\Gamma$.

Slowly oscillating coefficients. Let $(\Gamma, w) \in A_p^{SO}$. We denote by $SO(\Gamma)$ the set of all functions $c_\Gamma : \Gamma \to \mathbb{C}$ such that

$$c_\Gamma\big(t + re^{i\theta_k(r)}\big) = c_k(r) \quad \text{for } r \in \mathbb{R}_+ \text{ and } k = 1, 2, \dots, N,$$

where $c_k \in SO(\mathbb{R}_+)$. Below we assume that the coefficients of generalized singular integral operators with shifts are in $SO(\Gamma)$.

3. Mellin pseudodifferential operators

In this section we collect necessary results on Mellin pseudodifferential operators.

Let $V(\mathbb{R})$ be the set of all continuous functions $a : \mathbb{R} \to \mathbb{C}$ of bounded total variation $V(a)$ where

$$V(a) := \sup\Big\{\sum\nolimits_{k=1}^{n} \big|a(x_k) - a(x_{k-1})\big| : -\infty < x_0 < x_1 < \dots < x_n < +\infty,\ n \in \mathbb{N}\Big\}.$$

Hence (see, e.g., [7, Chapter 9]), there exist finite one-sided limits $a(\pm\infty) = \lim_{x\to\pm\infty} a(x)$, and therefore the function a is continuous on $\overline{\mathbb{R}} = [-\infty, +\infty]$. Clearly, $V(\mathbb{R})$ is a unital Banach algebra with the norm $\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a)$.

Let $C_b(\mathbb{R}_+, V(\mathbb{R}))$ be the set of all functions $a : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ such that $r \mapsto a(r,\cdot)$ is a bounded continuous $V(\mathbb{R})$-valued function on $\mathbb{R}_+$. Then the function $r \mapsto \|a(r,\cdot)\|_V$ belongs to $C_b(\mathbb{R}_+)$. The set $C_b(\mathbb{R}_+, V(\mathbb{R}))$ with the norm

$$\|a\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))} := \sup\big\{\|a(r,\cdot)\|_V : r \in \mathbb{R}_+\big\}$$

becomes a Banach algebra.

Let $d\mu(\varrho) = d\varrho/\varrho$ be the (normalized) invariant measure on $\mathbb{R}_+$ and let $C_0^\infty(\mathbb{R}_+)$ be the set of all infinitely differentiable functions of compact support on $\mathbb{R}_+$. By [13, Theorem 9.1] (also see [11, Theorem 3.1]), we have the following.

Theorem 3.1. *If $a \in C_b(\mathbb{R}_+, V(\mathbb{R}))$, then the Mellin pseudodifferential operator $OP(a)$, defined for functions $u \in C_0^\infty(\mathbb{R}_+)$ by the iterated integral*

$$\big[OP(a)u\big](r) = \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda \int_{\mathbb{R}_+} a(r,\lambda)\left(\frac{r}{\varrho}\right)^{i\lambda} u(\varrho)\,\frac{d\varrho}{\varrho}, \quad for\ \ r \in \mathbb{R}_+, \tag{3.1}$$

extends to a bounded linear operator on every Lebesgue space $L^p(\mathbb{R}_+, d\mu)$ with $1 < p < \infty$, and there is a number $C_p \in (0,\infty)$ depending only on p and such that

$$\|OP(a)\|_{\mathcal{B}(L^p(\mathbb{R}_+,d\mu))} \le C_p\,\|a\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))}.$$

By [12], a function $a \in C_b(\mathbb{R}_+, V(\mathbb{R}))$ is called slowly oscillating (at 0 and ∞) if $\lim\limits_{r\to s} cm_r^V(a) = 0$ for $s \in \{0,\infty\}$, where for some $\lambda > 1$ and all $r \in \mathbb{R}_+$,

$$cm_r^V(a) := \max\Big\{\big\|a(r,\cdot) - a(\nu r,\cdot)\big\|_V : \nu \in [\lambda^{-1},\lambda]\Big\}. \tag{3.2}$$

Clearly, the set $SO(\mathbb{R}_+, V(\mathbb{R}))$ of all slowly oscillating functions in $C_b(\mathbb{R}_+, V(\mathbb{R}))$ is a Banach subalgebra of $C_b(\mathbb{R}_+, V(\mathbb{R}))$. Obviously, for every $a \in SO(\mathbb{R}_+, V(\mathbb{R}))$, the $V(\mathbb{R})$-valued function $x \mapsto a(e^x,\cdot)$ is uniformly continuous on $\mathbb{R}$.

Let $a^h(r,\lambda) := a(r,\lambda+h)$ for all $(r,\lambda) \in \mathbb{R}_+ \times \mathbb{R}$. By [12, Section 2], the sets

$$\mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) := \Big\{a \in SO(\mathbb{R}_+, V(\mathbb{R})) : \lim_{|h|\to 0}\sup_{r\in\mathbb{R}_+}\big\|a(r,\cdot) - a^h(r,\cdot)\big\|_V = 0\Big\}, \tag{3.3}$$

$$\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) := \left\{a \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) : \lim_{M\to\infty}\sup_{r\in\mathbb{R}_+}\int_{\mathbb{R}\setminus[-M,M]} |\partial_\lambda a(r,\lambda)|\,d\lambda = 0\right\} \tag{3.4}$$

are Banach subalgebras of $SO(\mathbb{R}_+, V(\mathbb{R})) \subset C_b(\mathbb{R}_+, V(\mathbb{R}))$.

Below we need the following compactness results.

Theorem 3.2. [14, Theorems 3.4] *If $a(r,\lambda) \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ and $\lim\limits_{r^2+\lambda^2\to\infty} a(r,\lambda) = 0$, then the Mellin pseudodifferential operator $OP(a)$ is compact on every Lebesgue space $L^p(\mathbb{R}_+, d\mu)$ with $1 < p < \infty$.*

Theorem 3.3. [14, Theorem 3.5] *If $a(r,\lambda)$, $b(r,\lambda) \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, then the commutator $OP(a)OP(b) - OP(b)OP(a)$ is a compact operator on every Lebesgue space $L^p(\mathbb{R}_+, d\mu)$ with $1 < p < \infty$.*

4. Fredholm theory for the algebra $\mathfrak{A}_p$

Given $p \in (1,\infty)$, let $\mathcal{B}_p = \mathcal{B}(L^p(\mathbb{R}_+, d\mu))$, $\mathcal{K}_p = \mathcal{K}(L^p(\mathbb{R}_+, d\mu))$ and $\mathcal{B}_p^\pi = \mathcal{B}_p/\mathcal{K}_p$. Consider the Banach subalgebra $\mathfrak{A}_p$ of $\mathcal{B}_p$ generated by all Mellin pseudodifferential operators $OP(a)$ with symbols $a \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$, where the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is defined by (3.3)–(3.4). By [11, Lemma 10.1], for every $p \in (1,\infty)$, the Banach algebra $\mathfrak{A}_p$ contains all compact operators $K \in \mathcal{B}_p$. Consider the quotient algebra $\mathfrak{A}_p^\pi := \mathfrak{A}_p/\mathcal{K}_p$. By Theorem 3.3, $\mathfrak{A}_p^\pi$ is a commutative Banach algebra.

Let $\mathcal{M}_p$ denote the Banach algebra of all Fourier multipliers on $L^p(\mathbb{R})$, and let $C_p(\overline{\mathbb{R}})$ be the Banach subalgebra of $\mathcal{M}_p$ generated by the algebra $V(\mathbb{R})$. We also consider the Banach algebra $C_b\big(\mathbb{R}_+, C_p(\overline{\mathbb{R}})\big)$ of all bounded continuous $C_p(\overline{\mathbb{R}})$-valued functions $r \mapsto a(r,\cdot)$ equipped with the norm

$$\|a\|_{C_b(\mathbb{R}_+, C_p(\overline{\mathbb{R}}))} = \sup_{r\in\mathbb{R}_+} \|a(r,\cdot)\|_{\mathcal{M}_p}$$

and the Banach subalgebra $SO\big(\mathbb{R}_+, C_p(\overline{\mathbb{R}})\big)$ of $C_b\big(\mathbb{R}_+, C_p(\overline{\mathbb{R}})\big)$ that consists of all $C_p(\overline{\mathbb{R}})$-valued functions $r \mapsto a(r,\cdot)$ that slowly oscillate at 0 and ∞. The latter means that $\lim_{r\to s} cm_r^{\mathcal{M}_p}(a) = 0$ for $s \in \{0,\infty\}$, where similarly to (3.2),

$$cm_r^{\mathcal{M}_p}(a) := \max\Big\{\big\|a(r,\cdot) - a(\nu r,\cdot)\big\|_{\mathcal{M}_p} : \nu \in [\lambda^{-1}, \lambda]\Big\} \quad \text{for some } \lambda > 1.$$

Consider the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \subset SO(\mathbb{R}_+, V(\mathbb{R}))$ defined by (3.4). Since $V(\mathbb{R}) \subset C_p(\overline{\mathbb{R}})$, the closure $\widetilde{\mathcal{E}}_p$ of $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ in the norm of $C_b(\mathbb{R}_+, C_p(\overline{\mathbb{R}}))$ is a Banach subalgebra of $SO(\mathbb{R}_+, C_p(\overline{\mathbb{R}}))$. With every function $a \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ we associate the family $\widehat{a}$ which consists of the functions $a_\pm := a(\cdot, \pm\infty) \in SO(\mathbb{R}_+)$ and the functions $a_\xi := a(\xi,\cdot) \in V(\mathbb{R})$ (see [11, Lemma 2.7]) where $\xi \in \mathfrak{M} = M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+))$.

Let $\widehat{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ denote the set of all families $\widehat{a} = \big\{a_\pm,\, a_\xi : \xi \in \mathfrak{M}\big\}$ associated with the functions $a \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. We equip $\widehat{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ with usual operations of addition, multiplication, multiplication by scalars and the norm

$$\|\widehat{a}\|_p = \max\Big\{\|a_\pm\|_{C_b(\mathbb{R}_+)},\ \sup_{\xi\in\mathfrak{M}} \|a_\xi\|_{C_p(\overline{\mathbb{R}})}\Big\}. \tag{4.1}$$

Then $\widehat{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ becomes a non-closed commutative normed algebra. Let $\widehat{\mathfrak{A}}_p$ be the closure of $\widehat{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ in the norm (4.1). By analogy with [11], $\widehat{\mathfrak{A}}_p$ is a commutative Banach algebra of Fredholm symbols for the operators $A \in \mathfrak{A}_p$.

If $a \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$, then from [11, Lemma 11.9] it follows that for $s \in \{0,\infty\}$,

$$\limsup_{r\to s} \|a(r,\cdot)\|_{\mathcal{M}_p} = \sup_{\xi\in M_s(SO)} \|a(\xi,\cdot)\|_{\mathcal{M}_p}, \tag{4.2}$$

$$\liminf_{r\to s} \min_{\lambda\in\overline{\mathbb{R}}} |a(r,\lambda)| = \inf_{\xi\in M_s(SO)} \min_{\lambda\in\overline{\mathbb{R}}} |a(\xi,\lambda)|. \tag{4.3}$$

Since $C_p(\overline{\mathbb{R}})$ is the closure in $\mathcal{M}_p$ of the set $V_0(\mathbb{R})$ of all absolutely continuous functions in $V(\mathbb{R})$ (see [21, Lemma 1.1]), relations (4.2) and [11, Theorems 11.10 and 11.11] imply the following.

Theorem 4.1. *For each $p \in (1,\infty)$ there is a constant $\widetilde{C}_p \in (0,\infty)$ such that*

$$\|\widehat{a}\|_p \le \big\|[OP(a)]^\pi\big\| \le \widetilde{C}_p\,\|\widehat{a}\|_p \quad for\ every\ \ a \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})), \tag{4.4}$$

and hence the commutative Banach algebras $\mathfrak{A}_p^\pi$ and $\widehat{\mathfrak{A}}_p$ are isomorphic.

Because for any $A \in \mathfrak{A}_p$ there is a sequence $\{a_n\} \subset \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ such that $A = \lim\limits_{n\to\infty} OP(a_n)$ in $\mathfrak{A}_p$, we deduce from (4.4) that $\|\widehat{a}\|_p \le \|A^\pi\| \le C_p\,\|\widehat{a}\|_p$ for all $A \in \mathfrak{A}$, where $\widehat{a} = \lim\limits_{n\to\infty} \widehat{a}_n$ in the norm (4.1) and therefore $\widehat{a} \in \widehat{\mathfrak{A}}_p$. In particular, $K \in \mathcal{K}_p(\subset \mathfrak{A}_p)$ if and only if its Fredholm symbol in $\widehat{\mathfrak{A}}_p$ consists of zero functions.

On the other hand, applying [11, Lemma 10.2], we obtain the following.

Lemma 4.2. *For every $\widehat{a} \in \widehat{\mathfrak{A}}_p$ there is a function $a(r,\lambda) \in \widetilde{\mathcal{E}}_p \subset SO\big(\mathbb{R}_+, C_p(\overline{\mathbb{R}})\big)$ satisfying (4.2)–(4.3) and such that $\widehat{a} = \big\{a_\pm,\, a_\xi : \xi \in \mathfrak{M}\big\}$, where*

$$a_\pm := a(\cdot, \pm\infty) \in SO(\mathbb{R}_+), \quad a_\xi := a(\xi, \cdot) \in C_p(\overline{\mathbb{R}}).$$

Thus, by Lemma 4.2, to every operator $A \in \mathfrak{A}_p$ we may assign a function $a(r,\lambda) \in \widetilde{\mathcal{E}}_p \subset SO\big(\mathbb{R}_+, C_p(\overline{\mathbb{R}})\big)$ that forms the Fredholm symbol $\widehat{a} \in \widehat{\mathfrak{A}}_p$ of the operator $A \in \mathfrak{A}_p$. In that case we also write $A = OP(a)$.

Applying (4.2) and (4.3), we infer from [11, Section 12] the following Fredholm criterion and index formula for the operators $A \in \mathfrak{A}_p$.

Theorem 4.3. *A Mellin pseudodifferential operator $A \in \mathfrak{A}_p$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$ if and only if its Fredholm symbol $\widehat{a} \in \widehat{\mathfrak{A}}_p$ is invertible, that is,*

$$a(r, \pm\infty) \ne 0 \ \ for\ all\ \ r \in \mathbb{R}_+, \quad a(\xi,\lambda) \ne 0 \ \ for\ all\ \ (\xi,\lambda) \in \mathfrak{M} \times \overline{\mathbb{R}},$$

where $a(r,\lambda) \in \widetilde{\mathcal{E}}_p$ is a function associated with the Fredholm symbol $\widehat{a} \in \widehat{\mathfrak{A}}_p$ of the operator A. In the case of Fredholmness

$$\operatorname{Ind} A = \lim_{m\to\infty} \frac{1}{2\pi}\,\big\{\arg a(r,\lambda)\big\}_{(r,\lambda)\in\partial\Pi_m}$$

where $\Pi_m = [m^{-1}, m] \times \overline{\mathbb{R}}$ and $\big\{\arg a(r,\lambda)\big\}_{(r,\lambda)\in\partial\Pi_m}$ denotes the increment of $\arg a(r,\lambda)$ when the point (r,λ) traces the boundary $\partial\Pi_m$ of Π_m counter-clockwise.

By Theorem 3.3, [4, Theorem 1.14(c)] and the proof of [11, Theorem 12.5], Theorem 4.3 remains valid in the matrix case with $a(r,\lambda)$ replaced by $\det a(r,\lambda)$.

5. Applications of Mellin pseudodifferential operators

Let $1 < p < \infty$ and $(\Gamma, w) \in A_p^{SO}$. In this section we apply the results on Mellin pseudodifferential operators collected in Section 3 to generalized singular integral operators acting on the weighted Lebesgue space $L^p(\Gamma, w)$.

Given $1 < p < \infty$, we consider the Banach space $L^p_N(\mathbb{R}_+, d\mu)$ of vector functions $\varphi = \{\varphi_k\}_{k=1}^N$ with entries $\varphi_k \in L^p(\mathbb{R}_+, d\mu)$ and the norm

$$\|\varphi\| = \Big(\sum\nolimits_{k=1}^N \|\varphi_k\|^p_{L^p(\mathbb{R}_+,d\mu)}\Big)^{1/p}.$$

By analogy with [3], we introduce the isomorphisms

$$\Phi : L^p(\Gamma, w) \to L^p_N(\mathbb{R}_+, d\mu),\ (\Phi f)(r) = \{e^{v(r)} r^{1/p} f(t + re^{i\theta_k(r)})\}_{k=1}^N \ (r \in \mathbb{R}_+),$$
$$\Psi : \mathcal{B}(L^p(\Gamma, w)) \to \mathcal{B}(L^p_N(\mathbb{R}_+, d\mu)), \quad A \mapsto \Phi A \Phi^{-1}. \tag{5.1}$$

By (1.5) and the condition $(\Gamma, w) \in A_p^{SO}$, the operators $V_\alpha^{\pm 1}$, S_Γ and $V_\alpha S_\Gamma V_\alpha^{-1}$ are bounded on the space $L^p(\Gamma, w)$. As we will see, the Ψ-images of these operators to within compact operators are Mellin pseudodifferential operators of the form (3.1) with symbols in $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Theorem 5.1. [15, Theorems 5.3 and 5.4] *Let $1 < p < \infty$ and let $(\Gamma, w) \in A_p^{SO}$ and $\alpha \in \mathcal{A}_\Gamma$ be given by* (1.4), (2.6) *and* (2.7), *respectively, where for all $k = 1, 2, \dots, N$, the functions $\theta_{k+1} - \theta_k$ and ω_k are in $SO(\mathbb{R}_+)$ and*

$$(rD_r)^j \theta_k,\ (rD_r)^j v,\ (rD_r)^j \omega_k \in SO(\mathbb{R}_+) \quad for\ \ j = 1, 2, 3;$$
$$0 < \inf_{r, \varrho \in \mathbb{R}_+} \big(1/p + m_v(r, \varrho)\big) \le \sup_{r, \varrho \in \mathbb{R}_+} \big(1/p + m_v(r, \varrho)\big) < 1,$$

where $m_v(r, \varrho) := \big(v(r) - v(\varrho)\big)/(\ln r - \ln \varrho)$. If $c_\Gamma \in SO(\Gamma)$, then

$$\Psi(c_\Gamma I) = OP(c) = cI, \quad c(r) := \mathrm{diag}\{c_k(r)\}_{k=1}^N \ \ for\ r \in \mathbb{R}_+, \tag{5.2}$$

and the functions $c_k(r) \in SO(\mathbb{R}_+)$. For the operators $S_\Gamma, V_\alpha S_\Gamma V_\alpha^{-1} \in \mathcal{B}(L^p(\Gamma, w))$,

$$\Psi(S_\Gamma) = OP(\sigma) + K, \quad \Psi(V_\alpha S_\Gamma V_\alpha^{-1}) = OP(\sigma_\alpha) + K_\alpha,$$

where K and K_α are compact operators on the space $L^p_N(\mathbb{R}_+, d\mu)$,

$$\sigma(r, \lambda) := \big(\varepsilon_k\, \sigma_{j,k}(r, \lambda)\big)_{j,k=1}^N, \quad \sigma_\alpha(r, \lambda) := \big(\varepsilon_k\, \sigma_{j,k}^{(\alpha)}(r, \lambda)\big)_{j,k=1}^N, \tag{5.3}$$

$\varepsilon_k := 1$ if t is the starting point of the arc Γ_k and $\varepsilon_k := -1$ if t is the terminating point of Γ_k, the functions $\sigma_{j,k}(r, \lambda)$, $\sigma_{j,k}^{(\alpha)}(r, \lambda) \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ are given by

$$\sigma_{j,k}(r, \lambda) := \begin{cases} \exp\left[\Big(\pi\,\mathrm{sgn}(j-k) - \big(\theta_j(r) - \theta_k(r)\big)\Big) \dfrac{\lambda + i(1/p + rv'(r))}{1 + ir\theta'(r)}\right] \\ \times \left(\sinh\left[\pi \dfrac{\lambda + i(1/p + rv'(r))}{1 + ir\theta'(r)}\right]\right)^{-1} \quad \text{if}\ \ j \ne k, \\ \coth\left[\pi \dfrac{\lambda + i(1/p + rv'(r))}{1 + ir\theta'(r)}\right] \quad \text{if}\ \ j = k, \end{cases} \tag{5.4}$$

$$\sigma_{j,k}^{(\alpha)}(r,\lambda) := \begin{cases} \exp\left[\Big(\pi\,\mathrm{sgn}(j-k) - \big(\theta_j(r) - \theta_k(r)\big)\Big) \dfrac{\lambda + i(1/p + rv'(r))}{1 + ir\theta'(r)} \right. \\ \left. + i\big(\omega_j(r) - \omega_k(r)\big)\,[\lambda + i(1/p + rv'(r))]\right] \left(\sinh\left[\pi \dfrac{\lambda + i(1/p + rv'(r))}{1 + ir\theta'(r)}\right]\right)^{-1} \text{if}\ j \ne k, \\ \coth\left[\pi \dfrac{\lambda + i(1/p + rv'(r))}{1 + ir\theta'(r)}\right] \quad \text{if}\ j = k, \end{cases} \tag{5.5}$$

and $r\theta'(r) := r\theta_1'(r)$.

6. Banach algebra $\mathfrak{S}$ of generalized singular integral operators. Fredholmness and index

Given $p \in (1,\infty)$ and $(\Gamma, w) \in A_p^{SO}$, we intend to establish here a Fredholm criterion and an index formula for the operators in the Banach algebra $\mathfrak{S} \subset \mathcal{B}(L^p(\Gamma, w))$ generated by the operators $c_\Gamma I$ ($c_\Gamma \in SO(\Gamma)$) and $V_\alpha S_\Gamma V_\alpha^{-1}$ ($\alpha \in \mathcal{A}_\Gamma$).

Let $\widehat{\mathcal{D}}_{N\times N}$ be the Banach algebra of the families $\widehat{b} = \{b_\pm,\, b_\xi : \xi \in \mathfrak{M}\}$ where $b_\pm := b(\cdot, \pm\infty)$ are diagonal $N\times N$ matrix functions with entries $(b_\pm)_{i,i} \in SO(\mathbb{R}_+)$, $b_\xi := b(\xi, \cdot)$ are $N \times N$ matrix functions with entries $(b_\xi)_{i,j} \in C_p(\overline{\mathbb{R}})$ that vanishes at $\pm\infty$ if $i \neq j$, and the norm is given by

$$\|\widehat{b}\|_p = \max\Big\{ \max_{i=1,2,\dots,N} \big\|(b_\pm)_{i,i}\big\|_{C_b(\mathbb{R}_+)},\ N \sup_{\xi\in\mathfrak{M}} \max_{i,j=1,2,\dots,N} \big\|(b_\xi)_{i,j}\big\|_{C_p(\overline{\mathbb{R}})} \Big\}. \quad (6.1)$$

Let $\mathcal{A}_{N\times N}$ denote the algebra of $N \times N$ matrices with entries in an algebra $\mathcal{A}$.

By Theorem 5.1, the operators $\Psi(c_\Gamma I) = \mathrm{diag}\{c_k\}_{k=1}^N I$ for all $c_\Gamma \in SO(\Gamma)$ and $OP(\sigma_\alpha) = \Psi(V_\alpha S_\Gamma V_\alpha^{-1}) - K_\alpha$ for all $\alpha \in \mathcal{A}_\Gamma$ are Mellin pseudodifferential operators with matrix symbols in the Banach algebra $[\widetilde{\mathcal{E}}_p]_{N\times N} \subset [SO(\mathbb{R}_+, C_p(\overline{\mathbb{R}}))]_{N\times N}$, and their Fredholm symbols belong to the Banach algebra $\widehat{\mathcal{D}}_{N\times N}$.

Let $\widehat{\mathfrak{S}} := \widehat{\mathfrak{S}}_{p,\Gamma,w}$ be the Banach subalgebra of $\widehat{\mathcal{D}}_{N\times N}$ that consists of the Fredholm symbols $\widehat{b}$ of all Mellin pseudodifferential operators $OP(b)$ associated with the operators $\Psi(B)$ for $B \in \mathfrak{S}$. The latter means that $OP(b) - \Psi(B)$ are compact operators on the space $L_N^p(\mathbb{R}_+, d\mu)$, and hence B defines $\widehat{b} \in \widehat{\mathfrak{S}}$ uniquely, while $b \in [\widetilde{\mathcal{E}}_p]_{N\times N}$, in general, is not unique. In what follows we call the families $\widehat{b} \in \widehat{\mathfrak{S}}$ the Fredholm symbols of generalized singular integral operators $B \in \mathfrak{S}$.

Theorems 5.1 and 4.1 immediately imply the following.

Theorem 6.1. *Under the conditions of Theorem* 5.1*, let* Υ *be the mapping defined on the generators* $[c_\Gamma I]^\pi$ ($c_\Gamma \in SO(\Gamma)$) *and* $[V_\alpha S_\Gamma V_\alpha^{-1}]^\pi$ ($\alpha \in \mathcal{A}_\Gamma$) *of the Banach algebra* $\mathfrak{S}^\pi := \{B^\pi = B + \mathcal{K}(L^p(\Gamma, w)) : B \in \mathfrak{S}\}$ *by*

$$[c_\Gamma I]^\pi \mapsto \{c_\pm(\cdot),\ c(\xi) : \xi \in \mathfrak{M}\},$$
$$[V_\alpha S_\Gamma V_\alpha^{-1}]^\pi \mapsto \{\sigma_\alpha(\cdot, \pm\infty),\ \sigma_\alpha(\xi, \cdot) : \xi \in \mathfrak{M}\},$$

where $c_\pm(r) = c(r)$ *is defined by* (5.2), $\sigma_\alpha(r, \lambda) \in \big[\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))\big]_{N\times N}$ *is given by* (5.3) *and* (5.4) *if* $\alpha(t) = t$, *and by* (5.3) *and* (5.5) *if* $\alpha(t) \neq t$, *and* $\sigma_\alpha(\cdot, \pm\infty) = \mathrm{diag}\big\{\pm\, \varepsilon_k\big\}_{k=1}^N$. *Then* Υ *extends to the isomorphism*

$$\Upsilon :\ \mathfrak{S}^\pi \to \widehat{\mathfrak{S}}, \quad B^\pi \mapsto \widehat{b} = \{b(\cdot, \pm\infty),\, b(\xi, \cdot) : \xi \in \mathfrak{M}\},$$

where $B \in \mathfrak{S}$ *and their Fredholm symbols* $\widehat{b} \in \widehat{\mathfrak{S}}$ *are related by* $[OP(b)]^\pi = [\Psi(B)]^\pi$.

Theorem 6.1 and Theorem 4.3 in its matrix version imply the following.

Theorem 6.2. *If the conditions of Theorem* 5.1 *are fulfilled, then an operator* $B \in \mathfrak{S}$ *is Fredholm on the space* $L^p(\Gamma, w)$ *if and only if its Fredholm symbol* $\widehat{b}$ *is invertible* *in the algebra* $\widehat{\mathfrak{S}}$, *that is, for a matrix function* $b \in [\widetilde{\mathcal{E}}_p]_{N\times N}$ *associated with* $\widehat{b}$,

$$\begin{aligned} \det b(r, \pm\infty) \neq 0 \ \ \textit{for all} \ \ r \in \mathbb{R}_+, \\ \det b(\xi, \lambda) \neq 0 \ \ \textit{for all} \ \ (\xi, \lambda) \in \mathfrak{M} \times \overline{\mathbb{R}}, \end{aligned} \tag{6.2}$$

where $\mathfrak{M}$ *is defined by* (2.2). *In the case of Fredholmness*

$$\operatorname{Ind} B = \lim_{m\to+\infty} \frac{1}{2\pi} \big\{ \arg\det b(r,\lambda) \big\}_{(r,\lambda)\in\partial\Pi_m} \tag{6.3}$$

where $\Pi_m = [1/m, m] \times \overline{\mathbb{R}}$, $\min\big\{ |\det b(r,\lambda)| : (r,\lambda) \in \partial\Pi_m \big\} > 0$ *for all sufficiently large* $m > 0$, *and the point* (r, λ) *traces the boundary* $\partial\Pi_m$ *of* Π_m *counter-clockwise.*

7. Generalized singular integral operators with a Carleman shift

Let γ be a Carleman shift on Γ acting by the rule: $\gamma(\Gamma_k) = \Gamma_{k+1}$ for $k = 1, 2, \dots, N$ where $\Gamma_{N+1} = \Gamma_1$, the Nth iteration $\gamma_N = \gamma \circ \gamma_{N-1}$ of γ is the identity map, and for all $k = 1, 2, \dots, N$,

$$\gamma\big(t + re^{i\theta_k(r)}\big) = t + re^{\mu_k(r)} \exp\big(i\theta_{k+1}\big(re^{\mu_k(r)}\big)\big) \quad \text{ for } \ r \in \mathbb{R}_+, \tag{7.1}$$

where μ_k are real-valued functions in $C^3(\mathbb{R}_+)$ and the functions $(rD_r)^j \mu_k$ belong to $SO(\mathbb{R}_+)$ for all $j = 0, 1, 2, 3$. Hence the shift operator V_γ defined by $V_\gamma f = f \circ \gamma$ is bounded on the space $L^p(\Gamma, w)$, and $V_\gamma^N = I$.

By (7.1), the iterations γ_n of γ for $n = 0, 1, \dots, N-1$ are defined on Γ by

$$\gamma_n\big(t + re^{i\theta_k(r)}\big) = t + re^{\eta_{n,k}(r)} \exp\big(i\theta_{k+n}\big(re^{\eta_{n,k}(r)}\big)\big) \quad \text{ for } \ r \in \mathbb{R}_+,$$

where for all $n = 1, 2, \dots, N-1$ and all $k = 1, 2, \dots, N$,

$$\eta_{n,k}(r) = \eta_{n-1,k}(r) + \mu_{k+n}\big(re^{\eta_{n-1,k}(r)}\big) \quad \text{and} \quad \eta_{0,k}(r) = \eta_{N,k}(r) = 0, \tag{7.2}$$

$\theta_{k+n} = \theta_{k+n-N}$ and $\mu_{k+n} = \mu_{k+n-N}$ if $k + n > N$, and therefore all $\eta_{n,k}$ also are real-valued functions in $C^3(\mathbb{R}_+)$ and the functions $(rD_r)^j \eta_{n,k}$ belong to $SO(\mathbb{R}_+)$ for all $j = 0, 1, 2, 3$. Put $\eta_k(r) := \eta_{k,1}(r)$ for $r \in \mathbb{R}_+$ and all $k = 0, 1, \dots, N-1$.

Let us study the Banach subalgebra $\mathfrak{B} = \mathfrak{B}_{p,\Gamma,w,\gamma} \subset \mathcal{B}(L^p(\Gamma, w))$ of generalized singular integral operators with the shift γ. This algebra is generated by all operators $C \in \mathfrak{S}$ and by the shift operator $V_\gamma \in \mathcal{B}(L^p(\Gamma, w))$.

By analogy with (5.1), we introduce the isomorphisms

$$\begin{aligned} &\Phi_\gamma : L^p(\Gamma, w) \to L^p_N(\mathbb{R}_+, d\mu), \\ &(\Phi_\gamma f)(r) = \Big\{ e^{v(r)} r^{1/p} f\big(t + re^{\eta_{k-1}(r)} \exp\big[i\theta_k\big(re^{\eta_{k-1}(r)}\big)\big]\big) \Big\}_{k=1}^{N} \ \ (r \in \mathbb{R}_+), \\ &\Psi_\gamma : \mathcal{B}(L^p(\Gamma, w)) \to \mathcal{B}(L^p_N(\mathbb{R}_+, d\mu)), \quad A \mapsto \Phi_\gamma A \Phi_\gamma^{-1}. \end{aligned} \tag{7.3}$$

The isomorphism Ψ_γ allows us to eliminate the shift operators V_γ^n $(n>0)$ because

$$\Psi_\gamma(V_\gamma)=\Lambda_\gamma I \quad \text{with} \quad \Lambda_\gamma:=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{7.4}$$

Theorem 7.1. *If the conditions of Theorem* 5.1 *are fulfilled and* γ *is a Carleman shift on* Γ *defined above, then the operator* $\Psi_\gamma(V_\gamma)$ *is given by* (7.4) *and*

$$\Psi_\gamma(c_\Gamma I)=\operatorname{diag}\big\{c_k\big(re^{\eta_{k-1}(r)}\big)\big\}_{k=1}^N I, \quad \Psi_\gamma(V_\alpha S_\Gamma V_\alpha^{-1})=OP(\widehat{\sigma}_\alpha)+K_{\alpha,\gamma},$$

where $\eta_k(r)=\eta_{k,1}(r)$ *are given by* (7.2), $K_{\alpha,\gamma}\in\mathcal{K}(L_N^p(\mathbb{R}_+,d\mu))$,

$$\widehat{\sigma}_\alpha(r,\lambda):=\big(\varepsilon_k\,\widehat{\sigma}_{j,k}^{(\alpha)}(r,\lambda)\big)_{j,k=1}^N, \tag{7.5}$$

the functions $\widehat{\sigma}_{j,k}^{(\alpha)}(r,\lambda)\in\widetilde{\mathcal{E}}(\mathbb{R}_+,V(\mathbb{R}))$ *are given by*

$$\widehat{\sigma}_{j,k}^{(\alpha)}(r,\lambda):=$$
$$\begin{cases}\exp\Big[\Big(\pi\operatorname{sgn}(j-k)-\big(\theta_j(r)-\theta_k(r)\big)\Big)\dfrac{\lambda+i(1/p+rv'(r))}{1+ir\theta'(r)}\\ +i\big(\widehat{\omega}_j(r)-\widehat{\omega}_k(r)\big)\big[\lambda+i(1/p+rv'(r))\big]\Big]\Big(\sinh\Big[\pi\dfrac{\lambda+i(1/p+rv'(r))}{1+ir\theta'(r)}\Big]\Big)^{-1} & \text{if } j\neq k,\\ \coth\Big[\pi\dfrac{\lambda+i(1/p+rv'(r))}{1+ir\theta'(r)}\Big] \quad \text{if } j=k; \end{cases} \tag{7.6}$$

$$r\theta'(r):=r\theta_1'(r), \quad \widehat{\omega}_k(r):=\omega_k(r)+\eta_{k-1}(r). \tag{7.7}$$

Proof. Applying (7.3), we infer by analogy with [15, Theorem 5.3] that

$$\Big(\big[\Psi_\gamma(V_\alpha S_\Gamma V_\alpha^{-1})\big]_{j,k}f\Big)(r)$$
$$=\frac{\varepsilon_k}{\pi i}\int_{\mathbb{R}_+}\frac{e^{v(r)-v(\varrho)}(r/\varrho)^{1/p}\big(1+\varrho\widetilde{\omega}_k'(\varrho)+i\varrho\zeta_k'(\varrho)\big)e^{\widetilde{\omega}_k(\varrho)+i\zeta_k(\varrho)}}{\varrho e^{\widetilde{\omega}_k(\varrho)+i\zeta_k(\varrho)}-re^{\widetilde{\omega}_j(r)+i\zeta_j(r)}}f(\varrho)\,d\varrho,$$
$$=\frac{\varepsilon_k}{\pi i}\int_{\mathbb{R}_+}\frac{(r/\varrho)^{1/p+m_v(r,\varrho)}\big(1+\varrho\widetilde{\omega}_k'(\varrho)+i\varrho\zeta_k'(\varrho)\big)\,f(\varrho)}{1-\exp\big[i\big(\theta_j\big(\varrho e^{\widetilde{\omega}_k(\varrho)}\big)-\theta_k\big(\varrho e^{\widetilde{\omega}_k(\varrho)}\big)\big)\big]\big[\,(r/\varrho)\,e^{\widetilde{\omega}_j(r)-\widetilde{\omega}_k(\varrho)}\big]^{1+i\widetilde{\eta}_{j,k}(r,\varrho)}}\,\frac{d\varrho}{\varrho}$$
$$=\frac{1}{2\pi}\int_{\mathbb{R}}d\lambda\int_{\mathbb{R}_+}\varepsilon_k\,\frac{1+\varrho\widetilde{\omega}_k'(\varrho)+i\varrho\zeta_k'(\varrho)}{1+i\widetilde{\eta}_{j,k}(r,\varrho)}\,\widetilde{\sigma}_{j,k}^{(\alpha)}(r,\varrho,\lambda)\Big(\frac{r}{\varrho}\Big)^{i\lambda}f(\varrho)\frac{d\varrho}{\varrho},$$

where for $r\in\mathbb{R}_+$ and $k=1,2,\dots,N$,

$$\widetilde{\omega}_k(r):=\omega_k\big(re^{\eta_{k-1}(r)}\big)+\eta_{k-1}(r), \quad \zeta_k(r):=\theta_k\big(re^{\widetilde{\omega}_k(r)}\big), \tag{7.8}$$
$$\widetilde{\eta}_{j,k}(r,\varrho):=\big[\theta_j\big(re^{\widetilde{\omega}_j(r)}\big)-\theta_j\big(\varrho e^{\widetilde{\omega}_k(\varrho)}\big)\big]/\big[\ln r+\widetilde{\omega}_j(r)-\ln\varrho-\widetilde{\omega}_k(\varrho)\big],$$

$$\widetilde{\sigma}_\alpha(r,\varrho,\lambda) := \left(\varepsilon_k \frac{1+\varrho\widetilde{\omega}_k'(\varrho)+i\varrho\zeta_k'(\varrho)}{1+i\widetilde{\eta}_{j,k}(r,\varrho)}\, \widetilde{\sigma}_{j,k}^{(\alpha)}(r,\varrho,\lambda)\right)_{j,k=1}^N, \tag{7.9}$$

$$\widetilde{\sigma}_{j,k}^{(\alpha)}(r,\varrho,\lambda) :=$$

$$\begin{cases} \exp\Bigg[\Big(\pi\,\mathrm{sgn}\,(j-k) - \big(\theta_j\big(\varrho e^{\widetilde{\omega}_k(\varrho)}\big) - \theta_k\big(\varrho e^{\widetilde{\omega}_k(\varrho)}\big)\ \Big)\dfrac{\lambda+i(1/p+m_v(r,\varrho))}{1+i\widetilde{\eta}_{j,k}(r,\varrho)} \\ +i\big(\widetilde{\omega}_j(r)-\widetilde{\omega}_k(\varrho)\big)\big[\lambda+i(1/p+m_v(r,\varrho))\big]\Bigg]\left(\sinh\left[\pi\dfrac{\lambda+i(1/p+m_v(r,\varrho))}{1+i\widetilde{\eta}_{j,k}(r,\varrho)}\right]\right)^{-1} \\ \hfill \text{if } j\neq k, \\ \coth\left[\pi\dfrac{\lambda+i(1/p+m_v(r,\varrho))}{1+i\widetilde{\eta}_{j,k}(r,\varrho)}\right] \quad \text{if } \ j=k. \end{cases} \tag{7.10}$$

By analogy with [15, Theorem 5.4], we deduce from (7.9) and (7.10) that

$$\Psi_\gamma(V_\alpha S_\Gamma V_\alpha^{-1}) = OP(\breve{\sigma}_\alpha) + \widetilde{K}_{\alpha,\gamma},$$

where $\widetilde{K}_{\alpha,\gamma} \in \mathcal{K}(L_N^p(\mathbb{R}_+,d\mu))$, $\breve{\sigma}_\alpha(r,\lambda) := \big(\varepsilon_k\,\breve{\sigma}_{j,k}^{(\alpha)}(r,\lambda)\big)_{j,k=1}^N$ and $\breve{\sigma}_{j,k}^{(\alpha)}(r,\lambda)$ is given by (7.6) with $\widetilde{\omega}_j(r)$ and $\widetilde{\omega}_k(r)$ in place of $\widehat{\omega}_j(r)$ and $\widehat{\omega}_k(r)$, respectively. Comparing (5.5) and (7.6), we see that the functions $\breve{\sigma}_{j,k}^{(\alpha)}(r,\lambda)$ are in $\widetilde{\mathcal{E}}(\mathbb{R}_+,V(\mathbb{R}))$.

Finally, from (7.8) and (7.7) it follows that $\lim\limits_{r\to s}\big(\widetilde{\omega}_k(r)-\widehat{\omega}_k(r)\big)=0$ for all $k=0,1,\dots,N$ and all $s\in\{0,\infty\}$, whence $\lim\limits_{r^2+\lambda^2\to 0}\big[\breve{\sigma}_\alpha(r,\lambda)-\widehat{\sigma}_\alpha(r,\lambda)\big]=0$, which implies by Theorem 3.2 that the operator $OP(\breve{\sigma}_\alpha)-OP(\widehat{\sigma}_\alpha)$ is compact on the space $L_N^p(\mathbb{R}_+,d\mu)$. □

The following lemma gives a general form of operators in the algebra $\mathfrak{B}$.

Lemma 7.2. *Under the conditions of Theorem* 7.1*, every operator in the Banach algebra* $\mathfrak{B}$ *is a limit in* $\mathfrak{B}$ *of operators of the form*

$$B = B_0 + B_1V_\gamma + \cdots + B_{N-1}V_\gamma^{N-1},$$

where all operators B_k $(k=0,1,\dots,N-1)$ *belong to the Banach algebra* $\widetilde{\mathfrak{S}} := \widetilde{\mathfrak{S}}_{p,\Gamma,w,\gamma}$ *generated by the operators* $V_\gamma^n C V_\gamma^{-n}$ *with* $C\in\mathfrak{S}$ *and* $n=0,1,\dots,N-1$.

Let $\widetilde{\mathcal{D}}_{N\times N}$ denote the Banach algebra of the families $\widetilde{b}=\{b_\pm,\,b_\xi:\xi\in\mathfrak{M}\}$ where $b_\pm := b(\cdot,\pm\infty)$ are $N\times N$ matrix functions with entries $(b_\pm)_{i,j}\in SO(\mathbb{R}_+)$, $b_\xi := b(\xi,\cdot)$ are $N\times N$ matrix functions with entries $(b_\xi)_{i,j}\in C_p(\overline{\mathbb{R}})$, and the norm is given (cf. (6.1)) by

$$\|\widetilde{b}\|_p = N\max\Big\{\max_{i,j=1,2,\dots,N}\big\|(b_\pm)_{i,j}\big\|_{C_b(\mathbb{R}_+)},\ \sup_{\xi\in\mathfrak{M}}\max_{i,j=1,2,\dots,N}\big\|(b_\xi)_{i,j}\big\|_{C_p(\overline{\mathbb{R}})}\Big\}.$$

In view of Theorem 7.1, the operators $OP(\widehat{\sigma}_\alpha)=\Psi_\gamma(V_\alpha S_\Gamma V_\alpha^{-1})-K_{\alpha,\gamma}$ for $\alpha\in\mathcal{A}_\Gamma$, $\Psi_\gamma(c_\Gamma I)$ for $c_\Gamma\in SO(\Gamma)$, and $\Psi_\gamma(V_\gamma^n)=\Lambda_\gamma$ for $n=0,1,\dots,N-1$ are Mellin pseudodifferential operators $OP(b)$ with matrix symbols b in the Banach algebra $[\widetilde{\mathcal{E}}_p]_{N\times N}\subset[SO\big(\mathbb{R}_+,C_p(\overline{\mathbb{R}})\big)]_{N\times N}$, and their Fredholm symbols belong to the Banach algebra $\widetilde{\mathcal{D}}_{N\times N}$. Let $\widehat{\mathfrak{B}} := \widehat{\mathfrak{B}}_{p,\Gamma,w,\gamma}$ be the Banach subalgebra of $\widetilde{\mathcal{D}}_{N\times N}$ that consists of the Fredholm symbols $\widehat{b}$ of all Mellin pseudodifferential operators $OP(b)$ coinciding with $\Psi_\gamma(B)$ for $B \in \mathfrak{B}$ to within compact operators.

Since $\widehat{\sigma}_\alpha(r, \pm\infty) = \sigma_\alpha(r, \pm\infty)$, Theorems 7.1 and 4.1 imply the following.

Theorem 7.3. *Under the conditions of Theorem* 7.1*, let* Υ_γ *be the mapping defined on the generates* $[c_\Gamma I]^\pi$ $(c_\Gamma \in SO(\Gamma))$, $[V_\gamma]^\pi$ *and* $[V_\alpha S_\Gamma V_\alpha^{-1}]^\pi$ $(\alpha \in \mathcal{A}_\Gamma)$ *of the Banach algebra* $\mathfrak{B}^\pi$ *by*

$$[c_\Gamma I]^\pi \mapsto \big\{\widetilde{c}_\pm(\cdot),\, c(\xi) : \xi \in \mathfrak{M}\big\}, \quad [V_\gamma]^\pi \mapsto \big\{d_\pm,\, d_\xi : \xi \in \mathfrak{M}\big\},$$
$$[V_\alpha S_\Gamma V_\alpha^{-1}]^\pi \mapsto \big\{\sigma_\alpha(\cdot, \pm\infty),\, \widehat{\sigma}_\alpha(\xi, \cdot) : \xi \in \mathfrak{M}\big\},$$

where $\widetilde{c}_\pm(r) = \operatorname{diag}\big\{c_k\big(re^{\eta_{k-1}(r)}\big)\big\}_{k=1}^N$, $c(\xi) = \operatorname{diag}\big\{c_k(\xi)\big\}_{k=1}^N$, $d_\pm = d_\xi = \Lambda_\gamma$ *is given by* (7.4), $\sigma_\alpha(\cdot, \pm\infty) = \operatorname{diag}\big\{\pm\varepsilon_k\big\}_{k=1}^N$ *and* $\widehat{\sigma}_\alpha(r, \lambda) \in$ *is given by* (7.5)–(7.6). *Then* Υ_γ *extends to the isomorphism*

$$\Upsilon_\gamma : \mathfrak{B}^\pi \to \widehat{\mathfrak{B}}, \quad B^\pi \mapsto \widehat{b} = \big\{b(r, \pm\infty),\, b(\xi, \lambda) : \xi \in \mathfrak{M}\big\},$$

where $B \in \mathfrak{B}$ *and their Fredholm symbols* $\widehat{b} \in \widehat{\mathfrak{B}}$ *are related by* $[OP(b)]^\pi = [\Psi_\gamma(B)]^\pi$.

Finally, Theorem 7.3 and Theorem 4.3 in its matrix version give the following Fredholm result.

Theorem 7.4. *If the conditions of Theorem* 7.1 *are fulfilled, then an operator* $B \in \mathfrak{B}$ *is Fredholm on the space* $L^p(\Gamma, w)$ *if and only if its Fredholm symbol* $\widehat{b}$ *is invertible in the algebra* $\widehat{\mathfrak{B}}$ *or, equivalently,* (6.2) *holds for a matrix function* $b \in [\widetilde{\mathcal{E}}_p]_{N\times N}$ *associated with* $\widehat{b} \in \widehat{\mathfrak{B}}$. *If the operator* B *is Fredholm, then* $\operatorname{Ind} B$ *is given by* (6.3).

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Frozen History: Reconstructing the Climate of the Past

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Dedicated to Vladimir Maz′ya, a great mathematician and a great human being

Abstract. The ice caps on Greenland and Antarctica are huge memory banks: the temperature of the past is preserved deep in the ice. In this paper we present a mathematical model for the reconstruction of past temperatures from records of the present ones in a drilled hole.

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Keywords. Heat equation, ill-posed problem, trigonometric polynomial.

1. Introduction

The ice sheets on Greenland and Antarctica are huge memory banks. Dorthe Dahl-Jensen and collaborators (1998) measured the temperature in a hole in the ice cap of Greenland down to a depth of 3028.6 meters and could reconstruct the temperature on the surface of the ice during the last 50,000 years. The hole is situated at 72°35′ N, 37°38′ W and was drilled as a part of the Greenland Ice Core Project (GRIP).

The authors developed a Monte-Carlo method to fit the data and infer past climate. They made 3,300,000 forward calculations and chose the 2000 ones which gave the best fit to the temperatures recorded in the hole in 1995.

The climate of Antarctica over the longer period of 740,000 years was characterized through a new ice-core record. The core, drilled by the European Project for Ice Coring in Antarctica (EPICA 2004), came from Dome C (75°06′ S, 123°21′ E, altitude 3233 meters above sea level). The deuterium/hydrogen ratio was used as a temperature proxy, as in a more recent study of the newest ice core from Greenland, NGRIP (Steffensen et al. 2008). Thus the actual temperatures were not used in these cases. However, temperatures are routinely measured in the holes (Margareta Hansson, personal communication 2008-04-24).

The amplitude of the variations in the reconstructed temperatures from the GRIP hole varied between 23 K and 0.5 K and the precision of the measurements in the GRIP hole was 5 mK (Dahl-Jensen et al. 1998:268). Therefore some variations could be detected even at depths where the amplitude compared with the amplitude at the surface has been reduced by as much as a factor 500 or even 1000.

The purpose of the present paper is to analyze the feasibility of such a reconstruction. Since it involves solving the heat equation backwards, it is to be expected that the problem will be ill-posed. However, restricting attention to temperatures of a certain kind, a reasonably stable reconstruction is nevertheless possible.

We shall first investigate the continuity properties of a forward calculation, i.e., considering the surface temperature at all past moments as the given function and deducing the present temperature in a hole. Then we analyze the inverse operator yielding the past temperatures as a function of the present temperatures.

The main results of this paper were found in November 2002 when I was lecturing on partial differential equations at the Graduate School of Mathematics and Computing.

2. The heat equation

We consider

$$G = \{(t, x, y, z) \in \mathbf{R}^4;\ t \leqslant 0, z \leqslant \rho t\},$$

a sector in the four-dimensional space $\mathbf{R}^4$, as a model of the ice sheets on Greenland or Antarctica. Here t is the time, x and y are the horizontal coordinates, and z is the depth, counted negatively under the present ice surface.[1] The number ρ is a nonnegative constant allowing for the accumulation of snow. In fact, at the central parts of Greenland and Antarctica, the snow is kept in place, causing the surface to rise slowly. The lower strata flow out and become thinner but conserve their order. By contrast, at the periphery of the ice caps, the ice flows out into the surrounding oceans. The constant ρ is probably of the order of 10^{-9} m/s and can be taken to be zero for shorter periods.

In a more refined model, one should assume the horizontal dimensions of the ice to be bounded and restrict the depth to the range $-3028.6\,\mathrm{m} \leqslant z \leqslant 0$ in the case of Greenland. Also, one should consider the terrestrial heat flow from the underlying bedrock. However, in this first study we shall not do so; we also simplify and consider $-\infty < x, y < +\infty$ and $-\infty < z \leqslant \rho t$.

We consider *temperature functions*, by which we mean continuous complex-valued functions u on G which are of class C^2 in the interior G° of G and satisfy the heat equation

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{2.1}$$

[1] I apologize to geoscientists, who usually let z be positive below the surface.

there. Here κ is a positive constant, called *thermal diffusivity* or *temperature conduction capacity.* For ice at $-4\,^\circ\mathrm{C}$, $\kappa = 1.04 \cdot 10^{-6}\,\mathrm{m}^2/\mathrm{s}$. For copper this parameter is more than one hundred times larger, $1.1161 \cdot 10^{-4}\,\mathrm{m}^2/\mathrm{s}$, yielding an attenuation parameter β which is ten times smaller and the wavelength of the vertical function v ten times larger (see Section 7).

In our model we assume that the temperature functions are independent of x and y, so that we actually let G_1, defined as a sector in the third quadrant,

$$G_1 = \{(t, z) \in \mathbf{R}^2; t \leqslant 0, z \leqslant \rho t\},$$

be the model domain. The heat equation reduces to

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial z^2}. \tag{2.2}$$

For each temperature function in G_1 there is a function $h(t) = u(t, \rho t)$, $t \leqslant 0$, describing the temperature on the surface of the ice ($z = \rho t$) up to the present time, and a function $v(z) = u(0, z)$, $z \leqslant 0$, describing the present temperature in a hole in the ice ($t = 0$).[2] So we have restriction mappings $u \mapsto h$ and $u \mapsto v$. We now formulate two problems.

The direct problem. *Given a function on the surface of the ice for all past moments in time, find the present temperature at all depths $z \leqslant 0$. Thus, given $h(t) = u(t, \rho t)$ for $t \leqslant 0$, find $v(z) = u(0, z)$ for $z \leqslant 0$ for a suitable temperature function u.*

The inverse problem. *Given a function in a hole at the present time, find the temperature at the surface of the ice for all moments in the past. Thus, given $v(z) = u(0, z)$ for $z \leqslant 0$, find $h(t) = u(t, \rho t)$ for $t \leqslant 0$ for a suitable temperature function u.*

The boundary of G_1 consists of the two rays

$$S_1 = \{(t, \rho t) \in \mathbf{R}^2;\ t \leqslant 0\} \text{ and } S_2 = \{(0, z) \in \mathbf{R}^2;\ z \leqslant 0\},$$

thus $\partial G_1 = S_1 \cup S_2$. This means that the two problems are concerned with the boundary values of temperature functions in G_1 and whether one can pass from the values on one ray to the other.

3. Nonuniqueness in the direct problem

It is not possible to determine uniquely the present temperature v from the surface temperature h in the past: there are many temperatures u such that $h(t) = u(t, \rho t) = 0$ for all $t \leqslant 0$.

Indeed, the function $u(t, z) = Cz$ for $\rho = 0$ and $C\left(e^{\rho(\rho t - z)/\kappa} - 1\right)$ for $\rho \neq 0$ solves the equation with $h(t) = u(t, \rho t) = 0$ for any C. So there are infinitely many solutions, but the only bounded solution of this form is $u = 0$.

[2]Mnemonic trick: h for horizontal or history, v for vertical.

More generally, we may take

$$u(t,z) = C\big(e^{\kappa\beta^2 t+\beta z} - e^{\kappa\gamma^2 t+\gamma z}\big),$$

where β is an arbitrary number and $\gamma = -\beta - \rho/\kappa$. It is bounded only when it vanishes. When $\gamma = 0$ it reduces to the first-mentioned example.

4. Uniqueness in the direct problem

Proposition 4.1. *Given a function $h \in C(\mathbf{R}_-)$, there is at most one temperature function u in G_1 such that $u(t,\rho t) = h(t)$, $t \in \mathbf{R}_-$, in the class of temperature functions satisfying*

$$(-t_1)^{-1/2} \sup_{z \leqslant \rho t_1} |u(t_1,z)| \to 0 \text{ as } t_1 \to -\infty.$$

Here $\mathbf{R}_- = \{t \in \mathbf{R};\ t \leqslant 0\}$.

So the quantity $\sup_{z\leqslant\rho t_1} |u(t_1,z)|$ may grow slowly as $t_1 \to -\infty$, but it must be finite for all t_1, meaning that the solution must be bounded on each ray $\{(t_1,z) \in \mathbf{R}^2; z \leqslant \rho t_1\}$.

Proof. We know that for any solution U defined for $T_0 < t < T_1$ and all real z, a convolution formula enables us to calculate $U(t,\cdot)$ from $U(t_1,\cdot)$ when $T_0 < t_1 < t < T_1$:

$$U(t,\cdot) = E_{t-t_1} * U(t_1,\cdot),$$

where the convolution is with respect to the space variable only. Here $E_t(z) = (4\pi t)^{-1/2}\exp(z^2/(4t))$ is the heat kernel. Hence

$$U(t,z_1) - U(t,z_2) = \int_{\mathbf{R}} \big(E_{t-t_1}(z_1 - z) - E_{t-t_1}(z_2 - z)\big)U(t_1,z)dz.$$

We estimate the difference:

$$|U(t,z_1) - U(t,z_2)| \leqslant \sup_z |U(t_1,z)| \int \big|E_{t-t_1}(z_1 - z) - E_{t-t_1}(z_2 - z)\big|dz.$$

We now fix t and let t_1 tend to $-\infty$, assuming that $T_0 = -\infty$. The integral in the last formula is equal to a quantity $K(t - t_1, z_1 - z_2)$ which tends to zero as $t - t_1$ tends to $+\infty$; more precisely, it is bounded by a constant times $(-t_1)^{-1/2}$ for $t_1 \ll 0$. This shows that $|U(t,z_1) - U(t,z_2)| = 0$ as soon as $M(t_1) = \sup_z |U(t_1,z)|$ is bounded or more generally is such that $(-t_1)^{-1/2}M(t_1) \to 0$ as $t_1 \to -\infty$. Hence U is independent of z, so we have $U(t,z) = U(t,\rho t) = h(t)$ and conclude that U is determined by h.

However, all this supposes that the solutions are defined for all real z. If a solution u is only defined for $z \leqslant \rho t$ and vanishes for $z = \rho t$, one can extend it by mirroring: we define $U(t,z) = u(t,z)$ if $z \leqslant \rho t$ and $U(t,z) = -u(t, 2\rho t - z)V(t,z)$ if $z > \rho t$. Here we choose $V(t,z) = e^{At+Bz} = e^{\rho(\rho t - z)/\kappa}$, an exponential solution satisfying $V(t,z) = 1$ when $z = \rho t$. In fact, writing H for the heat operator $\partial/\partial t - \kappa\partial^2/\partial z^2$,

$$\begin{aligned}(HU)(t,z) &= -(Hu)(t,2\rho t - z)V(t,z) - 2u_z(t,2\rho t - z)(\rho V(t,z) + \kappa V_z(t,z)) \\ &\quad - u(t,2\rho t - z)(HV)(z,t), \qquad z > \rho t.\end{aligned}$$

Here $(Hu)(t,2\rho t - z) = (HV)(z,t) = 0$, and $\rho V + \kappa V_z$ vanishes with our choice of $A = \rho^2/\kappa$ and $B = -\rho/\kappa$. When $u(t,\rho t) = 0$, the extension is sufficiently smooth for the argument above to work; indeed the z-derivative of U from the right is equal to that of u from the left on the line $z = \rho t$. We note that $M(t_1) = \sup_{z\in\mathbf{R}} |U(t_1,z)| = \sup_{z\leqslant\rho t_1} |u(t_1,z)|$, for $|V(z,t_1)| \leqslant 1$ where $z \geqslant \rho t_1$, so the condition on the growth of u implies the same condition for the extension U.

If we have two solutions u_1 and u_2 with the same restriction $h(t) = u_1(t,\rho t) = u_2(t,\rho t)$, then we apply the above argument to $u_1 - u_2$. □

5. Nonuniqueness in the inverse problem

It is well known, thanks to A.N. Tihonov, that there is no uniqueness in the inverse problem; see, e.g., John (1991:211–213). However, two different solutions must differ by an unbounded function, indeed of very strong growth at infinity (Täcklind 1936).

6. Uniqueness in the inverse problem

We shall see that for the classes of functions we introduce, $\mathscr{V}$ and $\mathscr{H}$, we do have uniqueness in the inverse problem $v \mapsto h$.

7. Exponential solutions

An exponential function $u(t,z) = e^{At+Bz}$, where A and B are complex constants, is a solution to (2.2) if and only if $A = \kappa B^2$. The restriction of this function to the line $t = 0$ is the memory function $v(z) = u(0,z) = e^{Bz}$, and the restriction to the line $z = \rho t$ is the history $h(t) = u(t,\rho t) = e^{(A+\rho B)t}$.

Given a complex number α and real numbers β and γ we consider

$$u(t,z) = e^{(i\alpha - \rho(\beta+i\gamma))t + (\beta+i\gamma)z},$$

thus with $A = i\alpha - \rho(\beta + i\gamma)$ and $B = \beta + i\gamma$. This function satisfies

$$u(t,\rho t) = h(t) = e^{i\alpha t} \text{ and } u(0,z) = v(z) = e^{(\beta+i\gamma)z}.$$

It is a solution to the heat equation if and only if

$$\alpha = (2\kappa\beta + \rho)\gamma, \quad \gamma^2 = \beta^2 + \rho\beta/\kappa. \tag{7.1}$$

When $\rho = 0$ this simplifies to

$$\alpha = \pm 2\kappa\beta^2, \quad \gamma = \pm\beta. \tag{7.2}$$

In view of the application to temperatures, it is reasonable to assume that $h(t) = e^{i\alpha t}$ is bounded and does not tend to zero as $t \to -\infty$, which means that α should be real.

Given a nonnegative number β we therefore have two exponential solutions with damping like $e^{\beta z}$ as $z \to -\infty$ (coinciding and constant when $\beta = 0$):

$$u_1(t,z) = e^{i\alpha t+(\beta+i\gamma)z} \text{ for } \alpha \geqslant 0 \text{ and } u_2(t,z) = e^{i\alpha t+(\beta-i\gamma)z} \text{ for } \alpha \leqslant 0, \tag{7.3}$$

where α is given by (7.1). Their real and imaginary parts are, respectively

$$\operatorname{Re} u_1(t,z) = e^{\beta z}\cos(\alpha t+\gamma z), \quad \operatorname{Im} u_1(t,z) = e^{\beta z}\sin(\alpha t+\gamma z),$$
$$\operatorname{Re} u_2(t,z) = e^{\beta z}\cos(\alpha t-\gamma z), \quad \operatorname{Im} u_2(t,z) = e^{\beta z}\sin(\alpha t-\gamma z).$$

These functions have a *temporal period* $p = 2\pi/|\alpha|$, an *attenuation parameter* equal to $\beta \geqslant 0$ describing how the temperature tapers off as we go downwards, and a *spatial period* for the argument equal to $q = 2\pi/|\gamma|$.

When $\rho = 0$, (7.3) simplifies to

$$u_1(t,z) = e^{i\alpha t+\beta(1+i)z} \text{ for } \alpha \geqslant 0 \text{ and } u_2(t,z) = e^{i\alpha t+\beta(1-i)z} \text{ for } \alpha \leqslant 0, \tag{7.4}$$

where α is given by (7.2).

Conversely, given a real number α, there are unique numbers $\beta \geqslant 0$ and $|\gamma|$ such that (7.1) is satisfied. If $\rho = 0$, this simplifies to

$$\begin{aligned} &\beta = \sqrt{\frac{\alpha}{2\kappa}} \geqslant 0, \quad \gamma = \beta \geqslant 0, \text{ when } \alpha \geqslant 0, \text{ and} \\ &\beta = \sqrt{\frac{-\alpha}{2\kappa}} > 0, \quad \gamma = -\beta < 0, \text{ when } \alpha < 0. \end{aligned} \tag{7.5}$$

(The negative square roots will be disregarded.) In general β is the solution of an equation of degree four.

The quotients

$$\frac{|\alpha|}{2\kappa\beta^2} = \sqrt{1+\frac{\rho}{\kappa\beta}}\left(1+\frac{\rho}{2\kappa\beta}\right) \quad \text{and} \quad \frac{\gamma^2}{\beta^2} = 1+\frac{\rho}{\kappa\beta}$$

both increase with ρ when β is fixed, but the quotient between them,

$$\frac{q^2}{4\pi\kappa p} = \frac{|\alpha|}{2\kappa\gamma^2} = \frac{|\alpha|}{2\kappa\beta^2}\Big/\frac{\gamma^2}{\beta^2} = \frac{1}{|\gamma|}\left(\beta+\frac{\rho}{2\kappa}\right) = \frac{1+\frac{\rho}{2\kappa\beta}}{\sqrt{1+\frac{\rho}{\kappa\beta}}} = \sqrt{1+\frac{\rho^2}{4\kappa\beta(\kappa\beta+\rho)}}$$

does not vary so much with ρ for fixed β. Because of this the relation between the temporal period $p = 2\pi/|\alpha|$ and the spatial period $q = 2\pi/|\gamma|$ is not so sensitive for different values of ρ; on the other hand, given a temporal period p, the attenuation parameter β becomes smaller when ρ increases.

As an example we list the spatial periods for different temporal periods. We take $\rho = 0$, which in particular means that the attenuation parameter β is equal to $\gamma = 2\pi/q$. The numerical values in the table below will be different if $\rho > 0$, but the table will nevertheless give an idea of what we can expect. The attenuation over one half spatial period is $e^{-\pi} \approx 0.0432$ (at this depth, the variation is opposite to that at the surface); over one spatial period it is $e^{-2\pi} \approx 0.001867$; over one and a half spatial period $e^{-3\pi} \approx 0.0000807$. Thus the amplitude at the depth of one spatial period (where the variation is in phase with that at the surface) is just barely measurable.

At the depth of 1.06 meters the diurnal variation is reduced by a factor of $e^{2\pi} \approx 535$, but the amplitude of the annual variation is still quite large,

$$\exp(-2\pi/\sqrt{365.2422}) \approx 0.72.$$

Thus the variation due to a longer period in h is measurable at a greater depth even though, at the surface, the amplitudes of variations of shorter periods may be much larger. This is why the ice can serve as memory.

At the depths of

$$d_{0.01} = \frac{\log 100}{2\pi} q \approx 0.733q \text{ and } d_{0.001} = \frac{\log 1000}{2\pi} q \approx 1.10q$$

the amplitude of an oscillation is 1/100 and 1/1000, respectively, of the amplitude at the surface.

In the table below, we list various periods p of a wave $h(t) = e^{i\alpha t}$; its frequency $\alpha = 2\pi/p$ is measured in Hertz, $\mathrm{Hz} = \mathrm{s}^{-1}$, and the attenuation parameter $\beta = \sqrt{\alpha/(2\kappa)} \approx 1738/\sqrt{p}$ is expressed in inverse meters, m^{-1}. The thermal diffusivity κ is taken as $1.04 \cdot 10^{-6}\mathrm{m}^2/\mathrm{s}$. The last column gives the spatial period q of the argument, $q = 2\pi/\beta = \sqrt{4\pi\kappa p} \approx 0.003615\sqrt{p}$.

Temporal period p	$\alpha = 2\pi/p$	$\beta = \sqrt{\alpha/(2\kappa)}$	Spatial period $q = \sqrt{4\pi\kappa p}$
s	$\mathrm{Hz} = \mathrm{s}^{-1}$	m^{-1}	m
$24\,\mathrm{h} = 8.64 \cdot 10^4$	$7.27 \cdot 10^{-5}$	5.91	1.06
365.2422 days $= 3.1557 \cdot 10^7$	$1.99 \cdot 10^{-7}$	0.309	20.3
10 years $= 3.1557 \cdot 10^8$	$1.99 \cdot 10^{-8}$	$9.78 \cdot 10^{-2}$	64
100 years $= 3.1557 \cdot 10^9$	$1.99 \cdot 10^{-9}$	$3.09 \cdot 10^{-2}$	203
1000 years $= 3.1557 \cdot 10^{10}$	$1.99 \cdot 10^{-10}$	$9.78 \cdot 10^{-3}$	642
10,000 years $= 3.1557 \cdot 10^{11}$	$1.99 \cdot 10^{-11}$	$3.09 \cdot 10^{-3}$	2033
100,000 years $= 3.1557 \cdot 10^{12}$	$1.99 \cdot 10^{-12}$	$9.78 \cdot 10^{-4}$	6425

For instance, let us consider the diurnal variation of the surface temperature, thus with $\alpha = 2\pi/86{,}400\,\mathrm{s}^{-1}$. This gives $\beta \approx 5.91\,\mathrm{m}^{-1}$, which means that the amplitude at the surface is reduced by a factor of 535 at the depth of 1.06 meters.

For the annual variation we have

$$\alpha \approx 2\pi/(3.1557 \cdot 10^7)\,\mathrm{s}^{-1} \text{ and } \beta \approx 0.309\,\mathrm{m}^{-1},$$

meaning that the amplitude is reduced by a factor of 535 at the depth of 20.3 meters.

For the period of 80,000 years, which is perhaps typical for the ice ages, we get $\alpha = 2\pi/(3.15576\cdot 10^7\cdot 8\cdot 10^4) \approx 2.49\cdot 10^{-12}$ and $\beta \approx 0.001094\,\mathrm{m}^{-1}$, meaning that the amplitude is reduced by a factor of 23 at the depth of 2872 meters.

The fact that the spatial period is proportional to the square root of the temporal period is of course crucial, given the time period we want to study and the dimensions of the ice cap.

The amplitudes $b_{\beta+i\gamma}$ of the recorded temperatures (see (8.4)) varied in the interval $[0.25, 25]$ measured in kelvin, thus at most by a factor of 100, and the spatial frequencies β vary in the intervals indicated in the table. However, the diurnal and annual variations are negligible at depths larger than 20 meters, so the interesting periods are those from say 20 years to 50,000 years, yielding values for β in the interval $[0.0014, 0.07]$ measured in inverse meters, and spatial periods in the interval $[90, 4488]$ measured in meters.

8. Generalized trigonometric polynomials

We may combine the solutions found in section 7 as follows. The function

$$u(t,z) = \sum a_\alpha e^{(i\alpha-\rho(\beta+i\gamma))t+(\beta+i\gamma)z}, \qquad (t,z)\in\mathbf{R}^2, \tag{8.1}$$

where α, β and γ are related as in (7.1) and only finitely many of the coefficients a_α are nonzero, is a bounded solution of the heat equation, and the history and memory functions become

$$h(t) = \sum_{\alpha\in\mathbf{R}} a_\alpha e^{i\alpha t}, \quad t \leqslant 0; \qquad v(z) = \sum a_\alpha e^{(\beta+i\gamma)z}, \quad z\leqslant 0. \tag{8.2}$$

The function h is real valued if and only if $a_{-\alpha} = \overline{a_\alpha}$; in that case, $h(t) = a_0 + 2\sum_{\alpha>0}\operatorname{Re}(a_\alpha e^{i\alpha t}) = a_0 + 2\sum_{\alpha>0}\big(\operatorname{Re} a_\alpha\cos(\alpha t) - \operatorname{Im} a_\alpha \sin(\alpha t)\big)$.

The variation of the temperature over a couple of years may be described using just two periods, 24 hours and one year, or perhaps a little better with four periods, 12 hours, 24 hours, half a year and one year. To describe climate changes we need some longer periods, say from 20 years to 80,000 years.

Definition 8.1. Let $\mathscr{U}(G_1)$ or just $\mathscr{U}$ denote the space of all generalized trigonometric polynomials restricted to G_1,

$$u(t,z) = \sum_{\alpha\in\mathbf{R}} a_\alpha e^{i\alpha t+(\beta+i\gamma)z} = \sum_{\beta,\gamma\in\mathbf{R}} b_{\beta+i\gamma} e^{i\alpha t+(\beta+i\gamma)z}, \qquad (t,z)\in G_1,$$

where α, β, γ are related by (7.1) and the coefficients are related by $a_\alpha = b_{\beta+i\gamma}$ for these triples (α,β,γ). In the case when $\rho = 0$ we can have $b_{\beta+i\gamma}\neq 0$ only if $\gamma = \pm\beta$.

Here each attenuation parameter $\beta = \beta_\alpha$ is determined from α by (7.1), and all but finitely many of the coefficients $a_\alpha = b_{\beta+i\gamma}$ are zero. Given a function

$w\colon \mathbf{R} \to \,]0, +\infty[$, called the weight, we define the $\mathscr{U}$-norm of u as

$$\|u\|_{\mathscr{U}} = \sum_{\alpha} w(\alpha)|a_\alpha| = \sum_{\beta,\gamma} \tilde{w}(\beta + i\gamma)|b_{\beta+i\gamma}|.$$

Here

$$\tilde{w}(\beta + i\gamma) = w(\alpha) \text{ when } \alpha, \beta, \gamma \text{ are related by (7.1).} \tag{8.3}$$

It is sometimes convenient to assume, given the frequencies, that $\sum_\alpha w(\alpha) = \sum_{\beta,\gamma} \tilde{w}(\beta + i\gamma) = 1$, where the sum is over all real numbers α such that a_α is nonzero.

Definition 8.2. Let $\mathscr{H}(\mathbf{R}_-)$ or $\mathscr{H}$ denote the space of all trigonometric polynomials restricted to $\mathbf{R}_-$,

$$h(t) = \sum_{\alpha \in \mathbf{R}} a_\alpha e^{i\alpha t}, \qquad t \in \mathbf{R}_-,$$

where the α are real numbers and all but finitely many of the coefficients a_α are zero. We define the $\mathscr{H}$-norm of h as

$$\|h\|_{\mathscr{H}} = \sum_{\alpha \in \mathbf{R}} w(\alpha)|a_\alpha|.$$

Definition 8.3. Let $\mathscr{V}(\mathbf{R}_-)$ or $\mathscr{V}$ denote the space of all restrictions of generalized trigonometric polynomials to $\mathbf{R}_-$,

$$v(z) = \sum_{\beta,\gamma \in \mathbf{R}} b_{\beta+i\gamma} e^{(\beta+i\gamma)z}, \qquad z \in \mathbf{R}_-, \tag{8.4}$$

where all but finitely many of the coefficients $b_{\beta+i\gamma}$ are zero. We define the $\mathscr{V}$-norm of v as

$$\|v\|_{\mathscr{V}} = \sum_{\beta,\gamma} \tilde{w}(\beta + i\gamma)|b_{\beta+i\gamma}|,$$

where the weight $\tilde{w}$ is related to w by (8.3).

For the definitions of the norms to make sense, it is necessary to show that the coefficients are uniquely defined by the function values.

9. Finding the coefficients of a past temperature function

Proposition 9.1. *The coefficients of a function $h \in \mathscr{H}$ are given by the formula*

$$a_\alpha = \lim_{|I| \to +\infty} \frac{1}{|I|} \int_I h(t) e^{-i\alpha t} dt, \qquad \alpha \in \mathbf{R}. \tag{9.1}$$

Here $I = [r, s]$ is a subinterval of $\mathbf{R}_-$; its length $s - r$ is denoted by $|I|$ and assumed to be positive. This shows that $|a_\alpha| \leqslant \|h\|_\infty$.

Proof. We can find the coefficients a_θ, $\theta \in \mathbf{R}$, of h using the formula

$$\frac{1}{|I|}\int_I h(t)e^{-i\theta t}dt = a_\theta + \sum_{\alpha\neq\theta}\frac{a_\alpha}{|I|}\int_I e^{i(\alpha-\theta)t}dt, \qquad \theta \in \mathbf{R}.$$

Each of the terms after the first one tends to zero as $|I| \to +\infty$. □

10. Finding the coefficients of a memory function

Proposition 10.1. *For simplicity we consider now only the case $\rho = 0$. To find the coefficients $b_{\beta+i\gamma} = b_{\beta\pm i\beta}$ of a function $v \in \mathscr{V}$ we first extend it to the whole complex plane as an entire function w, thus $w(z)$ is given by the same formula for all $z \in \mathbf{C}$, and $w(z) = v(z)$ when $z \leqslant 0$. We then define*

$$v_1(z) = w\big(\tfrac{1}{2}(1+i)z\big) = \sum_{\beta\geqslant 0} b_{\beta+i\beta}e^{i\beta z} + \sum_{\beta>0} b_{\beta-i\beta}e^{\beta z}, \qquad z \in \mathbf{R}_-, \text{ and}$$

$$v_2(z) = w\big(\tfrac{1}{2}(1-i)z\big) = \sum_{\beta\geqslant 0} b_{\beta+i\beta}e^{\beta z} + \sum_{\beta>0} b_{\beta-i\beta}e^{-i\beta z}, \qquad z \in \mathbf{R}_-.$$

The coefficients are given by the formulas

$$b_{\theta+i\theta} = \lim_{s\to+\infty}\frac{1}{|I|}\int_I v_1(z)e^{-i\theta z}dz, \qquad \theta \geqslant 0, \text{ and}$$

$$b_{\theta-i\theta} = \lim_{|I|\to+\infty}\frac{1}{|I|}\int_I v_2(z)e^{i\theta z}dz, \qquad \theta > 0.$$

Here $I = [r, s]$ is an interval with $r < s \leqslant 0$.

Proof. We obtain

$$\begin{aligned}&\frac{1}{|I|}\int_I v_1(z)e^{-i\theta z}dz\\ &= b_{\theta+i\theta} + \sum_{\substack{\beta\geqslant 0\\ \beta\neq\theta}}\frac{1}{|I|}\int_I b_{\beta+i\beta}e^{i(\beta-\theta)z}dz + \sum_{\beta>0}\frac{1}{|I|}\int_I b_{\beta-i\beta}e^{(\beta-i\theta)z}dz.\end{aligned}$$

It is clear that all terms except the first tend to zero as $|I| \to +\infty$. Finding the coefficients $b_{\theta-i\theta}$ is similar. □

Another way to find the coefficients of a function $v \in \mathscr{V}$ is this:

Proposition 10.2. *First we note that $b_0 = \lim_{z\to-\infty} v(z)$. Assume that the smallest positive value of β for which $b_{\beta+i\beta}$ or $b_{\beta-i\beta}$ is nonzero is β_1. Then*

$$b_{\beta_1+i\beta_1} = \lim_{z\to-\infty}(v(z) - b_0)e^{-\beta_1(1+i)z} \quad \text{and} \quad b_{\beta_1-i\beta_1} = \lim_{z\to-\infty}(v(z) - b_0)e^{-\beta_1(1-i)z}.$$

Let β_2 be the smallest frequency after β_1. Then

$$b_{\beta_2+i\beta_2} = \lim_{z\to-\infty}\left(v(z) - b_0 - b_{\beta_1+i\beta_1}e^{\beta_1(1+i)z} - b_{\beta_1-i\beta_1}e^{\beta_1(1-i)z}\right)e^{-\beta_2(1+i)z},$$

$$b_{\beta_2-i\beta_2} = \lim_{z\to-\infty}\left(v(z) - b_0 - b_{\beta_1+i\beta_1}e^{\beta_1(1+i)z} - b_{\beta_1-i\beta_1}e^{\beta_1(1-i)z}\right)e^{-\beta_2(1-i)z}.$$

By repeated use of these formulas, all coefficients can be determined.

While the two last lemmas give the coefficients in the expansion of v theoretically, they are not suited for calculations. It is desirable to find a formula which gives at least approximately the values of the coefficients from information contained in an interval of finite length.

11. An isometry

Theorem 11.1. *The restriction mappings*

$$\mathscr{U} \ni u \mapsto u|_{\{(t,\rho t);t\in\mathbf{R}_-\}} = h \in \mathscr{H} \text{ and } \mathscr{U} \ni u \mapsto u|_{\{0\}\times\mathbf{R}_-} = v \in \mathscr{V}$$

are isometries.

Hence the mappings $\mathscr{H} \ni h \mapsto v \in \mathscr{V}$ and $\mathscr{V} \ni v \mapsto h \in \mathscr{H}$ are also isometries.

We thus have $\|u\|_{\mathscr{U}} = \|h\|_{\mathscr{H}} = \|v\|_{\mathscr{V}}$ if h and v are the restrictions of u. Therefore we also have isometries $h \mapsto v$ and $v \mapsto h$.

The problem is now moved over to a study of the norm $\|v\|_{\mathscr{V}}$ and a comparison between it and other norms. This we shall do in the next section.

We shall see in Theorems 14.1 and 15.1 that, if we use L^∞-norms, the problem $h \mapsto v$ is well posed under the topologies used and for generalized trigonometric polynomials h and v as long as the number of terms is bounded and the frequencies are well separated.

We may also estimate the L^1-norm of v in terms of the L^∞ norm of h. The constant term must be treated separately.

Theorem 11.2. *Let h and v be given as in Definitions* 8.2, 8.3 *and be related as in* (8.2)*. Then*

$$\|v - a_0\|_1 \leqslant C\|h - a_0\|_\infty, \tag{11.1}$$

where

$$C = \sqrt{2\kappa}\sum_{\alpha\neq0}|\alpha|^{-1/2};$$

the sum being extended over all the finitely many $\alpha \neq 0$ such that $a_\alpha \neq 0$.

Proof. We have

$$\|v - a_0\|_1 \leqslant \sum_{\beta\neq0}(|b_{\beta+i\beta}| + |b_{\beta-i\beta}|)\int_{-\infty}^0 e^{\beta z}dz = \sum_{\beta\neq0}(|b_{\beta+i\beta|} + |b_{\beta-i\beta}|)\beta^{-1}.$$

In view of the fact that $|b_{\beta+i\beta}| + |b_{\beta-i\beta}| = |a_\alpha|$, that $\beta^{-1} = \sqrt{2\kappa/|\alpha|}$, and that $|a_\alpha| \leqslant \|h - a_0\|_\infty$ for $\alpha \neq 0$, the last expressing is not larger than

$$\sum_{\alpha\neq 0} |a_\alpha|\beta^{-1} \leqslant \|h - a_0\|_\infty \sqrt{2\kappa} \sum_{\alpha\neq 0} |\alpha|^{-1/2},$$

where the sum is extended over the finitely many real numbers α such that $\alpha \neq 0$ and $a_\alpha \neq 0$. □

12. The direct problem

Theorem 12.1. *Given a temperature history $h \in \mathscr{H}$ of the temperature $h(t)$, $t \leqslant 0$, at the surface for all past moments in time, there is a unique memory $v = \mathbf{M}(h) \in \mathscr{V}$ of that history giving the temperature at the present time at all depths in the hole.*

The memory $\mathbf{M}(h)$ is obtained in the following way. First the set of coefficients $(a_\alpha)_\alpha$ of h is determined by formula (9.1), and then the coefficients $(b_{\beta+i\gamma})_{\beta,\gamma}$ using the relations (7.1). Finally v is synthesized as (8.4). The mapping $\mathbf{M}\colon \mathscr{H} \to \mathscr{V}$ is an isometry. We can then combine it with the injection $J\colon \mathscr{V} \to L^\infty(\mathbf{R}_-)$, which is of norm one. The mapping $J \circ \mathbf{M}\colon \mathscr{H} \to L^\infty(\mathbf{R}_-)$ is linear, injective, and continuous.

We shall see that the norm $\|\cdot\|_{\mathscr{H}}$ used in $\mathscr{H}$ is equivalent to the L^∞-norm over a bounded interval under the hypotheses stated in Theorem 14.1.

13. The inverse problem

Theorem 13.1. *Given data $v \in \mathscr{V}$ of the temperature $v(z)$, $z \leqslant 0$, at all depths in the hole at the present time, there is a unique past $h = \mathbf{P}(v) \in \mathscr{H}$ giving the temperature $h(t)$ at the surface of the ice at all past moments in time.*

The history $\mathbf{P}(v)$ is obtained in the following way. First the two functions v_1 and v_2 are determined using analytic continuation. Then the indexed families of coefficients $(b_{\beta+i\beta})_\beta$ and $(b_{\beta-i\beta})_\beta$ are determined, as well as the family of coefficients $(a_\alpha)_\alpha$. Finally h is synthesized as in (8.2). The mapping $\mathbf{P}\colon \mathscr{V} \to \mathscr{H}$ is the inverse of $\mathbf{M}$ and thus also an isometry.

There are three difficulties in the model constructed so far.

The first difficulty in the determination of $\mathbf{P}(v)$ lies in the analytic continuation giving v_1, v_2 from v. All other steps in the construction are well defined mappings which have reasonable continuity properties. Analytic continuation, on the other hand, is a notoriously ill-posed problem. We have concentrated all the difficulties of the inverse problem into the mappings $v \mapsto v_1, v_2$.

Second, the norms defined require the functions to be known over the unbounded interval $\mathbf{R}_-$. It is desirable to replace them by norms using only values in a bounded subinterval I.

Third, in applications it is necessary to find a good approximation $v \in \mathscr{V}$ to given temperature measurements $V(z_1), V(z_2), \ldots, V(z_P)$ at finitely many points z_p, $p = 1, 2, \ldots, P$.

Concerning the inverse problem $v \mapsto h$, that of reconstructing the temperature in the past from knowledge of the temperature in the hole at present, we may ask whether it is possible to turn (11.1) around and obtain an inequality like

$$\|h\|_\infty \leqslant C\|v\|_1 \tag{13.1}$$

could hold. But this is not so: take $h(t) = e^{i\alpha t}$, $v(z) = e^{\beta(1+i)z}$. Then $\|h\|_\infty = 1$ while $\|v\|_1 = 1/\beta \to 0$ as $\alpha \to +\infty$. This shows that (13.1) does not hold. However, this leaves open the possibility that (13.1) could hold if we assume that the frequencies α which build up h are bounded.

14. Other norms for the space of past temperature functions

We have seen that there is an isometry $h \mapsto v$ between the past temperature functions $h \in \mathscr{H}$ and the memory functions $v \in \mathscr{V}$. While the norm in $\mathscr{H}$ is rather natural, we have used in the space $\mathscr{V}$ a rather elusive norm. The purpose of the present section and the next one is to compare these norms with more easily accessible ones.

Example. We note that

$$h_{\alpha,\varepsilon}(t) = \frac{1}{\varepsilon}e^{i(\alpha+\varepsilon)t} - \frac{1}{\varepsilon}e^{i\alpha t}, \qquad t \in \mathbf{R}_-,\ \alpha \in \mathbf{R},\ \varepsilon \neq 0,$$

is a trigonometric polynomial of norm $\|h_{\alpha,\varepsilon}\|_{\mathscr{H}} = \|h_{\alpha,\varepsilon}\|_\infty = 2/\varepsilon$, $\varepsilon \neq 0$. However, for every bounded subinterval $I = [r, s]$ of $\mathbf{R}_-$ with $r < s \leqslant 0$, the norm $\|h_{\alpha,\varepsilon}|_I\|_\infty$ tends to a finite limit as ε tends to zero, viz. $|r|$. Indeed, $h_{\alpha,\varepsilon} \to ite^{i\alpha t}$, an exponential polynomial, as $\varepsilon \to 0$, uniformly when t is bounded. □

The example shows that there is no estimate $\|h\|_{\mathscr{H}} \leqslant C\|h|_I\|_\infty$ for any bounded interval I, and also that the class $\mathscr{H}$ is not closed under uniform convergence on bounded sets. To make it a closed subspace of $C(\mathbf{R}_-)$ we would have to consider exponential polynomials, i.e., exponential functions with polynomials as coefficients, just as in the theory of ordinary differential equations. However, if we keep the frequencies apart, the situation is different:

Theorem 14.1. *For all functions $h \in \mathscr{H}$, $h(t) = \sum a_\alpha e^{i\alpha t}$, $t \in \mathbf{R}_-$, we have an estimate*

$$\|h\|_\infty \leqslant A\|h\|_{\mathscr{H}}, \quad \textit{where} \quad A = \Big(\inf_\alpha w(\alpha)\Big)^{-1}.$$

Here the infimum is taken over the finitely many α such that $a_\alpha \neq 0$.

Conversely, if the interval $I \subset \mathbf{R}_-$ is long enough and the frequencies are kept apart, then

$$\|h\|_{\mathscr{H}} \leqslant C\|h|_I\|_\infty$$

for some constant C; more precisely

$$(1-c)\|h\|_{\mathscr{H}} \leqslant \sum_\theta w(\theta)\|h|_I\|_\infty, \text{ where } c = \frac{1}{|I|}\sum_\theta w(\theta)\sup_{\alpha\neq\theta}\frac{2}{w(\alpha)|\alpha-\theta|}.$$

If $|I|$ is so large that $c<1$, we obtain an estimate. Here the supremum and the sum are taken only over those frequencies α and θ such that the coefficients a_α, a_θ do not vanish.

If in particular $w(\alpha)=1$ for all α, then it is enough that $|I|$ be larger than $2n/\gamma$, where

$$\gamma = \inf_{\alpha,\theta,\alpha\neq\theta}|\alpha-\theta| > 0,$$

and where n is the number of frequencies α such that $a_\alpha \neq 0$.

For applications it is of course important that we have the supremum over a bounded interval I rather than all of $\mathbf{R}_-$ in this estimate. So in the set $\mathscr{H}_{n,\gamma}$ of all functions in $\mathscr{H}$ with at most n frequencies and $|\alpha-\theta| \geqslant \gamma > 0$ when $\alpha \neq \theta$, the two norms are equivalent. However, $\mathscr{H}_{n,\gamma}$ is not a vector space.

For the proof of the theorem we shall need the following lemma.

Lemma 14.2. *Let us define an entire function φ by $\varphi(\zeta) = \int_{-1}^0 e^{\zeta t}dt$; equivalently*

$$\varphi(\zeta) = \begin{cases} \dfrac{1-e^{-\zeta}}{\zeta}, & \zeta \in \mathbf{C} \smallsetminus \{0\}; \\ 1, & \zeta = 0. \end{cases}$$

Then

$$|\varphi(\zeta)| \leqslant \frac{1}{\operatorname{Re}\zeta}, \qquad \zeta \in \mathbf{C}, \operatorname{Re}\zeta > 0, \text{ and}$$

$$|\varphi(\zeta)| \leqslant \frac{1+e^{-\operatorname{Re}\zeta}}{|\zeta|}, \qquad \zeta \in \mathbf{C} \smallsetminus \{0\}.$$

In particular $|\varphi(i\eta)| \leqslant 2/|\eta|$, and, since $|\varphi(i\eta)| \leqslant 1$,

$$|\varphi(i\eta)| \leqslant \min\left(1, \frac{2}{|\eta|}\right)$$

with equality when $\eta = 0$ or $\eta \in \pi(2\mathbf{Z}+1)$.

The function φ appears in a natural way when we calculate the mean values of exponential functions. For any interval $I = [r,s]$ of length $|I| = s-r > 0$ and all complex numbers ζ we have

$$\frac{1}{|I|}\int_I e^{\zeta t}dt = \frac{1}{s-r}\int_r^s e^{\zeta t}dt = e^{\zeta s}\varphi(|I|\zeta).$$

The estimates in the lemma are easily proved.

Proof of Theorem 14.1. The first part of the theorem is easy to prove: we have

$$|h(t)| \leqslant \sum_\alpha |a_\alpha| = \sum_\alpha w(\alpha)|a_\alpha|w(\alpha)^{-1} \leqslant \|h\|_{\mathscr{H}} \sup_\alpha w(\alpha)^{-1}.$$

For the second part we take the mean value of $h(t)e^{-\theta t}$ over an interval $I = [r, s] \subset \mathbf{R}_-$ to be determined later. This gives

$$M_{I,\theta} = \frac{1}{|I|}\int_I h(t)e^{-i\theta t}dt = a_\theta + \sum_{\alpha\neq\theta} a_\alpha e^{i(\alpha-\theta)s}\frac{1-e^{-i|I|(\alpha-\theta)}}{i|I|(\alpha-\theta)}$$
$$= a_\theta + \sum_{\alpha\neq\theta} a_\alpha e^{i(\alpha-\theta)s}\varphi(i|I|(\alpha-\theta)),$$

where $\varphi\colon \mathbf{C} \to \mathbf{C}$ is the function in the lemma.

We may now estimate

$$|M_{I,\theta} - a_\theta| \leqslant \sum_{\alpha\neq\theta} |a_\alpha \varphi(i|I|(\alpha-\theta))| = \sum_{\alpha\neq\theta} w(\alpha)|a_\alpha| \frac{\varphi(i|I|(\alpha-\theta))}{w(\alpha)} \leqslant \frac{c_\theta}{|I|}\|h\|_{\mathscr{H}}$$

if we define

$$c_\theta = |I| \sup_{\alpha\neq\theta} \frac{\varphi(i|I|(\alpha-\theta))}{w(\alpha)} \leqslant \sup_{\alpha\neq\theta} \frac{2}{w(\alpha)|\alpha-\theta|},$$

so that

$$|a_\theta| \leqslant |M_{I,\theta}| + |M_{I,\theta} - a_\theta| \leqslant |M_{I,\theta}| + c_\theta|I|^{-1}\|h\|_{\mathscr{H}} \leqslant \|h|_I\|_\infty + c_\theta|I|^{-1}\|h\|_{\mathscr{H}},$$

leading to

$$w(\theta)|a_\theta| \leqslant w(\theta)\|h|_I\|_\infty + c_\theta|I|^{-1}w(\theta)\|h\|_{\mathscr{H}}.$$

Summing over all θ we obtain

$$\|h\|_{\mathscr{H}}\left(1 - \frac{1}{|I|}\sum_\theta c_\theta w(\theta)\right) \leqslant \left(\sum_\theta w(\theta)\right)\|h|_I\|_\infty. \qquad \square$$

Example. Let us take four frequencies, say with temporal periods $p_j = 2000$, 4000, 8000, 16,000 years, corresponding to frequencies $\alpha_j = 2\pi/p_j = 2\pi/2000$, $2\pi/4000$, $2\pi/8000$, $2\pi/16{,}000$ years^{-1}. Then

$$\frac{1}{\gamma} = \sup_{j\neq k}\frac{1}{|\alpha_j - \alpha_k|} = \frac{1}{2\pi}\sup_{j\neq k}\frac{p_jp_k}{p_j - p_k} = \frac{16{,}000}{2\pi} \text{ years},$$

so that an interval of length $|I| > 2n/\gamma \approx 20{,}372$ years suffices to define an equivalent norm.

15. Other norms for the space of memory functions

We would like to have a result like Theorem 14.1 also for the memory functions, thus to estimate the $\mathscr{V}$-norm by the supremum norm over a bounded interval. This problem is subtler because of the presence of the damping factors $e^{\beta z}$. As an example we present the following result, which however needs a strong separation of the attenuation parameters. There is probably a trade-off between the length of the interval needed in an estimate and the separation of the attenuation parameters which has to be assumed; the following result is just an instance of this.

Theorem 15.1. *For any function $v \in \mathscr{V}$ of the form*

$$v(z) = b_0 + \sum_1^n \left(b_j e^{(\beta_j + i\beta_j)z} + c_j e^{(\beta_j - i\beta_j)z} \right), \qquad z \leqslant 0,$$

with attenuation parameters $\beta_0 = 0 < \beta_1 < \cdots < \beta_n$ and coefficients $b_0, \ldots, b_n$, $c_1, \ldots, c_n$, we have an estimate

$$\|v\|_\infty \leqslant \|v\|_{\mathscr{V}}$$

if we use the weights $w_j = 1$ for all j. Conversely, assume that the function is real valued, so that $b_0 \in \mathbf{R}$ and $c_j = \overline{b}_j$, and that the parameters are well separated in the sense that $\beta_j \leqslant \sigma\beta_{j+1}$ for some number $\sigma < \frac{1}{3}$, or, a little weaker, $\sigma < \frac{1}{3}(1 + 2\sigma^n)$. Then

$$\|v - b_0\|_{\mathscr{V}} \leqslant \frac{1}{\cos\theta} \|v|_I - b_0\|_\infty,$$

where $\theta = \sigma\pi(1 - \sigma^{n-1})/(1 - \sigma) < \pi/2$ and I is the interval $I = [s_n, 0]$ with

$$s_n = -\frac{2\pi}{\beta_1}\frac{1 - \sigma^n}{1 - \sigma} \geqslant -\frac{2\pi}{\beta_1}\frac{1}{1 - \sigma} > -\frac{3\pi}{\beta_1},$$

and the weights are defined by

$$w_j = e^{\beta_j s_n}.$$

From this weight we can pass to other weights.

Proof. The first statement is easy to prove. For the second, we may assume that $b_0 = 0$.

We note that, given any $\theta \in \,]0, \pi/2[$, the inequality $\operatorname{Re}\zeta \geqslant (\cos\theta)|\zeta|$ holds for complex numbers ζ when $\zeta/|\zeta|$ lies on an arc on the unit circle of length 2θ, occupying a fraction $\tau = \theta/\pi < \frac{1}{2}$ of the whole circumference. Using this idea, we see that a term

$$b_j e^{(\beta_j + i\beta_j)z} + c_j e^{(\beta_j - i\beta_j)z} = 2e^{\beta_j z} \operatorname{Re}\left(b_j e^{i\beta_j z}\right)$$

is at least equal to $2\cos\theta\, e^{\beta_j z}|b_j|$ on a union of intervals $M_j = \bigcup_{k \in \mathbf{Z}} M_j(k)$, each $M_j(k)$ being of length $2\pi\tau/\beta_j = \tau q_j$ and appearing periodically with a period of $q_j = 2\pi/\beta_j$, thus $M_j(k) = M_j(0) + kq_j$, $k \in \mathbf{Z}$. Here $q_j = 2\pi/\beta_j$ are the spatial periods, satisfying $q_1 > q_2 > \cdots > q_n$ and $\sigma q_{j-1} \geqslant q_j$.

The idea is to find a point $s_n \leqslant 0$ in the intersection $\bigcap_j M_j$. Then the interval $I = I_n = [s_n, 0]$, in fact even $[s_n, s_n]$, can serve in our conclusion.

Define $s_0 = 0$ and let s_1 be the right endpoint of the interval $M_1(k_1)$ which is contained in $\mathbf{R}_-$ but such that $M_1(k_1 + 1)$ is not contained in $\mathbf{R}_-$. Thus $s_0 - q_1 < s_1 \leqslant s_0$.

Suppose we have already found $s_0, s_1, \ldots, s_{j-1}$ such that $s_{j-2} - q_{j-1} < s_{j-1} \leqslant s_{j-2}$. Then we define s_j as the right endpoint of an interval $M_j(k_j) = [r_j, s_j]$ such that $s_j \leqslant s_{j-1}$ and $s_j + q_j > s_{j-1}$. Thus $s_{j-1} - q_j < s_j \leqslant s_{j-1}$, establishing the induction step.

The right endpoints s_j of the intervals $M_j(k_j) = [r_j, s_j] = [s_j - \tau q_j, s_j]$ form a decreasing sequence by construction.

We shall estimate s_n. We have

$$\begin{aligned} s_n &> s_{n-1} - q_n > s_{n-2} - q_{n-1} - q_n > \cdots \\ &> s_1 - q_2 - q_3 - \cdots - q_n \geqslant s_1 - q_2(1 + \sigma + \sigma^2 + \cdots + \sigma^{n-2}) \\ &= s_1 - q_2 \frac{1 - \sigma^{n-1}}{1 - \sigma} \geqslant -q_1 \frac{1 - \sigma^n}{1 - \sigma}. \end{aligned}$$

We observe that $r_1 = s_1 - \tau q_1$, the left endpoint of the first interval $[r_1, s_1]$ lies to the left of s_n if we choose τ properly. Indeed,

$$s_n - r_1 = s_n - s_1 + \tau q_1 > -q_2 \frac{1 - \sigma^{n-1}}{1 - \sigma} + \tau q_1 = \frac{1 - \sigma^{n-1}}{1 - \sigma}(-q_2 + \sigma q_1) \geqslant 0$$

if we choose $\tau = \sigma(1 - \sigma^{n-1})/(1 - \sigma)$.

The point $z = s_n$ is therefore a point where all inequalities

$$b_j e^{(\beta_j + i\beta_j)z} + c_j e^{(\beta_j - i\beta_j)z} \geqslant 2 \cos\theta \, e^{\beta_j z} |b_j| = \cos\theta \, e^{\beta_j z}(|b_j| + |c_j|)$$

hold.

We now have

$$\|v|_I\|_\infty \geqslant \operatorname{Re} v(s_n) = 2 \sum_1^n e^{\beta_j s_n} \operatorname{Re}\left(b_j e^{i\beta_j s_n}\right) \geqslant 2\cos\theta \sum_1^n e^{\beta_j s_n} |b_j| = \cos\theta \|v\|_{\mathscr{V}},$$

where we have used the weights $w_j = e^{\beta_j s_n}$, $j = 1, \ldots, n$.

Remark 15.2. If $\sigma < \frac{1}{3}$ we may choose $\tau = \sigma/(1 - \sigma) < \frac{1}{2}$ and can prove that the left endpoints r_j form an increasing sequence, so that the intervals form a decreasing sequence of sets, and the final interval $[r_n, s_n]$ is contained in all of them.

We may estimate $r_j - r_{j-1}$ as follows.

$$\begin{aligned} r_j - r_{j-1} &= s_j - \tau q_j - s_{j-1} + \tau q_{j-1} \\ &> -q_j - \tau q_j + \tau q_{j-1} = \tau q_{j-1} - (1 + \tau) q_j \geqslant 0, \end{aligned}$$

since now $\tau = (1 + \tau)\sigma$. This proves that $[r_j, s_j] \subset [r_{j-1}, s_{j-1}]$. □

Example. Let the longest spatial period be $q_1 = 2000$ meters, corresponding to a temporal period $p_1 = 4\pi\kappa q_1^2 \approx 3.06 \cdot 10^{11}$ seconds ≈ 9699 years. Then with a separation parameter $\sigma = 0.3$ we get $\tau = \sigma(1 - \sigma) = 3/7 \approx 0.42857$ and $I = [s_n, 0]$ with $s_n \geqslant -1.429 q_1 = 2858$ meters for any number of terms.

16. What remains to be done?

This paper presents just the first steps in an investigation of a model for the reconstruction of past temperatures. A sensitivity analysis should be performed, as well as an investigation of a corresponding discrete model.

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Multidimensional Harmonic Functions Analogues of Sharp Real-part Theorems in Complex Function Theory

Gershon Kresin

Dedicated to Vladimir Maz'ya on the occasion of his 70th birthday

Abstract. In the present paper, the sharp multidimensional analogues of Lindelöf inequality and similar estimates for analytic functions are considered. Using a sharp inequality for the gradient of a bounded or semibounded harmonic function in a ball, one arrives at improved estimates (compared with the known ones) for the gradient of harmonic functions in an arbitrary subdomain of $\mathbb{R}^n$. A representation of the sharp constant in a pointwise estimate of the gradient of a harmonic function in a half-space is obtained under the assumption that function's boundary values belong to L^p. This representation is realized in the three-dimensional case and the values of sharp constants are explicitly given for $p = 1, 2, \infty$.

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1. Introduction

Among different sharp pointwise estimates in the theory of analytic functions of one variable, there are estimates with the real part of an analytic function in their majorant parts. Various inequalities of such a type are called *real-part theorems* in reference to the first assertion of such a kind, the celebrated Hadamard real-part theorem

$$|f(z)| \leq \frac{C|z|}{1-|z|} \max_{|\zeta|=1} \Re f(\zeta).$$

Here $|z| < 1$ and f is an analytic function on the closure $\overline{\mathbb{D}}$ of the unit disk $\mathbb{D} = \{z : |z| < 1\}$ vanishing at $z = 0$. This inequality was first obtained by

Hadamard with $C = 4$ in 1892 [7]. The following refinement of Hadamard real-part theorem due to Borel [3, 4], Carathéodory [13, 14] and Lindelöf [15]

$$|f(z) - f(0)| \le \frac{2|z|}{1-|z|} \sup_{|\zeta|<1} \Re\{f(\zeta) - f(0)\}, \tag{1.1}$$

and corollaries of the last sharp estimate are often called the Borel-Carathéodory inequalities. Sometimes, (1.1) is called Hadamard-Borel-Carathéodory inequality (e.g., Burckel [5]). We note that an extension of Hadamard's real-part theorem for holomorphic functions in a subdomain of a complex manifold was established by Aizenberg, Aytuna and Djakov [1].

Real-part theorems embrace a family of sharp estimates for derivative of arbitrary analytic function. Among them, there is the Lindelöf inequality [15] (see also Jensen [9])

$$|f'(z)| \le \frac{2}{1-|z|^2} \sup_{|\zeta|<1} \{\Re f(\zeta) - \Re f(z)\}, \tag{1.2}$$

where z is an arbitrary point of the unit disk $\mathbb{D}$ and f is an analytic function on $\mathbb{D}$ with real part bounded from above. There exists also the analogous sharp estimate (e.g., Ingham [8], Rajagopal [16]), containing $\Re f(0)$ on the right-hand side instead of $\Re f(z)$:

$$|f'(z)| \le \frac{2}{(1-|z|)^2} \sup_{|\zeta|<1} \{\Re f(\zeta) - \Re f(0)\}. \tag{1.3}$$

Furthermore, we mention the Ruscheweyh inequality [17]

$$|f'(z)| \le \frac{2}{1-|z|^2} \Re f(z) \tag{1.4}$$

for analytic functions with positive real part on $\mathbb{D}$.

There is a series of sharp estimates for derivative of functions f analytic in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ with different characteristics of the real part of f in the majorant part (see, e.g., [10]). In particular, it is the Lindelöf inequality in the half-plane

$$|f'(z)| \le \frac{1}{\Im z} \sup_{\zeta \in \mathbb{C}_+} \{\Re f(\zeta) - \Re f(z)\} \tag{1.5}$$

and two equivalent inequalities

$$|f'(z)| \le \frac{2}{\pi \Im z} \sup_{\zeta \in \mathbb{C}_+} |\Re f(\zeta)| \tag{1.6}$$

and

$$|f'(z)| \le \frac{1}{\pi \Im z} \mathcal{O}_{\Re f}(\mathbb{C}_+), \tag{1.7}$$

where $\mathcal{O}_{\Re f}(\mathbb{C}_+)$ is the oscillation of $\Re f$ on $\mathbb{C}_+$, and z is an arbitrary point in $\mathbb{C}_+$. The inequalities for the first derivative of an analytic function just mentioned can be restated as estimates for the gradient of a harmonic function. For example, Lindelöf's inequality (1.2) can be written in the form

$$|\nabla u(z)| \leq \frac{2}{1-|z|^2} \sup_{|\zeta|<1} \{u(\zeta) - u(z)\}, \tag{1.8}$$

where u is a harmonic function in the unit disk.

The collection of real-part theorems, related assertions and their generalizations on multidimensional cases is rather broad. It involves assertions of various form (see, e.g., [10] and the bibliography collected there).

The present article is devoted to analogues of sharp estimates (1.2)–(1.7) for harmonic functions in multidimensional domains.

In the first part we state multidimensional generalizations of (1.2)–(1.5) for the gradients of bounded or semibounded harmonic functions. In particular, we present pointwise estimates with sharp coefficients for the gradient of bounded or semibounded harmonic functions in a multidimensional ball. As a corollary, we obtain similar sharp estimates in a half-space, as well as improved estimates (in comparison with the known ones) for the gradient of harmonic functions in an arbitrary domain $G \subset \mathbb{R}^n$.

The second part of the present paper relates the joint work with V. Maz'ya. We give a representation for the sharp constant in an inequality for the gradient of a harmonic function in a half-space assuming that function's boundary value belong L^p. Special case of the three-dimensional situation is discussed. In particular, we present sharp constants for analogues of real-valued counterparts of estimates (1.6) and (1.7).

2. Pointwise estimates for the gradient of bounded or semibounded harmonic functions in multidimensional domains

The sharp inequality for $|\nabla u(x)|$ in Proposition below is parameter dependent. The role of a parameter is played by a number $\lambda \in [0,\ 1]$. A parametric sharp real-part theorems were treated in [10, 12].

Introduction of the parameter enables one to unify several variants of the right-hand side. Each variant goes back to a corresponding analogue in the theory of analytic functions, that is to one of sharp real-part theorems.

We introduce some notation used henceforth. Let $|\cdot|$ be the length of a vector in the n-dimensional Euclidean space $\mathbb{R}^n$. Let, further, $\mathbb{B}_\rho = \big\{x \in \mathbb{R}^n : |x| < \rho\big\}$. We put $\mathbb{B} = \mathbb{B}_1$.

The proof of the statement given below will be published in Functional Differential Equations, vol. 16, 2009.

Proposition 2.1. *Let x be an arbitrary point in $\mathbb{B}$, and let $0 \le \lambda \le 1$. Then, for any harmonic function u on $\mathbb{B}$, the inequality*

$$|\nabla u(x)| \le \frac{\Big(n+(n-2)|x|\Big)(1-\lambda|x|)^{n-1}}{(1-|x|)^n(1+\lambda|x|)}\,\mathcal{B}_{\lambda x}(u)$$

holds with the sharp coefficient in front of $\mathcal{B}_{\lambda x}(u)$. Here $\mathcal{B}_{\lambda x}(u)$ is either of the following expressions:

(i) $\sup\limits_{y\in\mathbb{B}} u(y) - u(\lambda x)$,

(ii) $u(\lambda x) - \inf\limits_{y\in\mathbb{B}} u(y)$,

(iii) $\sup_{y\in\mathbb{B}} |u(y)| - |u(\lambda x)|$.

In particular,

$$|\nabla u(x)| \le \frac{n+(n-2)|x|}{1-|x|^2}\,\mathcal{B}_x(u), \tag{2.1}$$

and

$$|\nabla u(x)| \le \frac{n+(n-2)|x|}{(1-|x|)^n}\,\mathcal{B}_0(u). \tag{2.2}$$

Remark 2.1. The variant (i) of inequality (2.1) in Proposition 2.1

$$|\nabla u(x)| \le \frac{n+(n-2)|x|}{1-|x|^2}\sup_{|y|<1}\{u(y)-u(x)\},$$

is an extension of the Lindelöf estimate (1.8) to the n-dimensional case.

Remark 2.2. Similarly, the variant (i) of estimate (2.2) in Proposition 2.1

$$|\nabla u(x)| \le \frac{n+(n-2)|x|}{(1-|x|)^n}\sup_{|y|<1}\{u(y)-u(0)\},$$

is a multidimensional generalization of the Hadamard-Borel-Carathéodory type inequality (1.3) for the derivative of an analytic function.

Remark 2.3. Now, suppose that a harmonic function u is positive on $\mathbb{B}$. Then, by the variant (ii) of (2.1) and (2.2) in Proposition 2.1 we have

$$|\nabla u(x)| \le \frac{n+(n-2)|x|}{1-|x|^2}\,u(x), \qquad |\nabla u(x)| \le \frac{n+(n-2)|x|}{(1-|x|)^n}\,u(0),$$

respectively. The first of these estimates is an extension to the multidimensional case of the Ruscheweyh inequality (1.4) written for the gradient of a harmonic function. Note that the estimate $|\nabla u(0)| \le n\,u(0)$ follows directly from the classical Harnack inequality in the unit ball.

Remark 2.4. The variant (iii) of inequality (2.2) in Proposition 2.1

$$|\nabla u(x)| \le \frac{n+(n-2)|x|}{(1-|x|)^n}\sup_{|y|<1}\{\,|u(y)|-|u(0)|\},$$

is a multidimensional generalization of one of E. Landau's type inequalities for the first derivative of an analytic function (see Kresin and Maz'ya [10], p. 70, pp. 81–85).

By dilation, we obtain the following statement, equivalent to Proposition 2.1 and involving the ball $\mathbb{B}_R$ with an arbitrary R.

Proposition 2.2. *Let x be an arbitrary point in $\mathbb{B}_R$, and let $0 \le \lambda \le 1$. Then, for any harmonic function on $\mathbb{B}_R$, the inequality*

$$|\nabla u(x)| \le \frac{\Big(nR + (n-2)|x|\Big)(R - \lambda|x|)^{n-1}}{(R-|x|)^n (R+\lambda|x|)} \, \mathcal{B}_{\lambda x, R}(u)$$

holds with the sharp coefficient in front of $\mathcal{B}_{\lambda x,R}(u)$. Here $\mathcal{B}_{\lambda x,R}(u)$ is either of the following expressions:

(i) $\sup\limits_{y\in\mathbb{B}_R} u(y) - u(\lambda x)$,

(ii) $u(\lambda x) - \inf\limits_{y\in\mathbb{B}_R} u(y)$,

(iii) $\sup\limits_{y\in\mathbb{B}_R} |u(y)| - |u(\lambda x)|$.

In particular,

$$|\nabla u(x)| \le \frac{nR + (n-2)|x|}{R^2 - |x|^2} \, \mathcal{B}_{x,R}(u), \tag{2.3}$$

and

$$|\nabla u(x)| \le \frac{nR + (n-2)|x|}{R^{2-n}(R-|x|)^n} \, \mathcal{B}_{0,R}(u).$$

Now, we formulate two corollaries of Proposition 2.2. The first of them contains, in particular, an analogue of the real formulation of inequality (1.5) for an open half-space $\mathbb{H}$ in $\mathbb{R}^n$.

We note first that with the notation $d_x = R - |x|$, one can write (2.3) in the form

$$|\nabla u(x)| \le \frac{nR + (n-2)(R-d_x)}{d_x(2R-d_x)} \, \mathcal{B}_{x,R}(u). \tag{2.4}$$

Passing to the limit as $R \to \infty$ in (2.4) so that $d_x = \text{const}$, we arrive at the following assertion.

Corollary 2.1. *Let x be an arbitrary point in an open half-space $\mathbb{H} \subset \mathbb{R}^n$, and let $d_x = \text{dist}(x, \partial\mathbb{H})$. Then, for any harmonic function on $\mathbb{H}$, the inequality*

$$|\nabla u(x)| \le \frac{n-1}{d_x} \, \mathcal{H}_x(u)$$

holds with the sharp coefficient in front of $\mathcal{H}_x(u)$. Here $\mathcal{H}_x(u)$ is either of the following expressions:

(i) $\sup\limits_{y\in\mathbb{H}} u(y) - u(x)$,

(ii) $u(x) - \inf\limits_{y\in\mathbb{H}} u(y)$,

(iii) $\sup\limits_{y\in\mathbb{H}} |u(y)| - |u(x)|$.

The second corollary of Proposition 2.2 contains an improvement of the interior estimate

$$|\nabla u(x)| \le \frac{n}{d_x} \sup_{y\in G} \big\{u(y) - u(x)\big\} \tag{2.5}$$

and similar estimates (see, e.g., Gilbarg and Trudinger [6], Ch. 2) for the gradient of a harmonic function u on a domain $G \subset \mathbb{R}^n$. Here $x \in G$, and $d_x = \text{dist}(x, \partial G)$.

The improvement just mentioned of the estimates for the gradient of a harmonic function in G stems directly from (2.4). It is formulated in the next assertion.

Corollary 2.2. *Let x be a fixed point in a domain $G \subset \mathbb{R}^n$, and let ξ_x be a point on ∂G for which $|\xi_x - x| = d_x$. Further, let R be the radius of the largest ball lying entirely in G with center on the straight line L passing through x and ξ_x. Then, for any harmonic function u on G, the inequality*

$$|\nabla u(x)| \le \frac{nR + (n-2)(R - d_x)}{d_x(2R - d_x)}\, \mathcal{G}_x(u) \tag{2.6}$$

holds, where $\mathcal{G}_x(u)$ is either of the following expressions:

(i) $\sup\limits_{y\in G} u(y) - u(x)$,

(ii) $u(x) - \inf\limits_{y\in G} u(y)$,

(iii) $\sup\limits_{y\in G} |u(y)| - |u(x)|$.

We note that the factor

$$\frac{nR + (n-2)(R - d_x)}{2R - d_x}$$

before $1/d_x$ in (2.6) tends to $n-1$, as $x \to \xi_x,\ x \in L$, whereas the similar coefficient in (2.5) is equal to the constant n.

The next section contains joint results with V. Maz'ya. Our paper [11] concerns sharp pointwise estimates for solutions of strongly elliptic second-order systems with boundary data from L^p.

3. Sharp pointwise estimates for the gradient of harmonic functions in the half-space with boundary data from L^p

We introduce some notation used henceforth. Let $x \in \mathbb{R}^n_+ = \big\{x = (x', x_n) : x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\big\}$ and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. By $|| \cdot ||_p$ we denote the norm in the space $L^p(\mathbb{R}^{n-1})$, that is

$$||f||_p = \left\{ \int_{\mathbb{R}^{n-1}} |f(x')|^p \, dx' \right\}^{1/p},$$

if $1 \le p < \infty$, and $||f||_\infty = \text{ess sup}\{|f(x')| : x' \in \mathbb{R}^{n-1}\}$. Further, by $h^p(\mathbb{R}^n_+)$ we denote the Hardy space of harmonic functions on $\mathbb{R}^n_+$, which can be represented as the Poisson integral

$$u(x) = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{x_n}{|y-x|^n} \, u(y')dy' \tag{3.1}$$

with boundary values in $L^p(\mathbb{R}^{n-1})$, $1 \le p \le \infty$, where ω_n is the area of the unit sphere in $\mathbb{R}^n$ and $y = (y', 0)$ (compare, for example, with [2]) .

Proposition 3.1. *Let $u \in h^p(\mathbb{R}^n_+)$, and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_p(x)$ in the inequality*

$$|\nabla u(x)| \le \mathcal{C}_p(x) \big\| u|_{x_n=0} \big\|_p \tag{3.2}$$

is given by

$$\mathcal{C}_p(x) = C_p \, x_n^{(1-n-p)/p}, \tag{3.3}$$

where

$$C_1 = \frac{2(n-1)}{\omega_n}, \tag{3.4}$$

and

$$C_p = \frac{2}{\omega_n} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}_-} F_{n,p}(\sigma, \boldsymbol{z}) \, d\sigma \right\}^{(p-1)/p} \tag{3.5}$$

for $1 < p \le \infty$. Here

$$F_{n,p}(\sigma, \boldsymbol{z}) = \big| \big(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma, \ \boldsymbol{z} \big) \big|^{p/(p-1)} \big(\boldsymbol{e}_\sigma, -\boldsymbol{e}_n \big)^{n/(p-1)}, \tag{3.6}$$

$\boldsymbol{z} \in \mathbb{R}^n$, and $\boldsymbol{e}_\sigma$ is the n-dimensional unit vector joining the origin to a point σ on the lower hemisphere $\mathbb{S}^{n-1}_- = \{x \in \mathbb{R}^n : |x| = 1, x_n < 0\}$.

Proof. Let $x = (x', x_n)$ be a fixed point in $\mathbb{R}^n_+$. The representation (3.1) implies

$$\frac{\partial u}{\partial x_i} = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \left[\frac{\delta_{ni}}{|y-x|^n} + \frac{n x_n (y_i - x_i)}{|y-x|^{n+2}} \right] u(y')dy',$$

that is

$$\begin{aligned} \nabla u(x) &= \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \left[\frac{\boldsymbol{e}_n}{|y-x|^n} + \frac{n x_n (y-x)}{|y-x|^{n+2}} \right] u(y')dy' \\ &= \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy}}{|y-x|^n} \, u(y')dy', \end{aligned}$$

where $\boldsymbol{e}_{xy} = (y-x)|y-x|^{-1}$. For any $\boldsymbol{z} \in \mathbb{R}^n$,

$$(\nabla u(x), \boldsymbol{z}) = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy}, \ \boldsymbol{z})}{|y-x|^n} \, u(y')dy'$$

and therefore,

$$\left|\nabla u(x)\right| = \frac{2}{\omega_n} \sup_{|\boldsymbol{z}|=1} \int_{\mathbb{R}^{n-1}} \frac{(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{z})}{|y-x|^n}\ u(y')dy'.$$

Hence, by the permutation property of suprema,

$$\mathcal{C}_p(x) = \frac{2}{\omega_n} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{R}^{n-1}} \frac{\left|(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy}, \boldsymbol{z})\right|^q}{|y-x|^{nq}}\ dy' \right\}^{1/q} \tag{3.7}$$

for $1 < p \le \infty$, where $p^{-1} + q^{-1} = 1$, and

$$\begin{aligned} \mathcal{C}_1(x) &= \frac{2}{\omega_n} \sup_{|\boldsymbol{z}|=1} \sup_{y \in \mathbb{R}^{n-1}} \frac{|(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{z})|}{|y-x|^n} \\ &= \frac{2}{\omega_n} \sup_{y \in \mathbb{R}^{n-1}} \frac{|\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy}|}{|y-x|^n}. \end{aligned} \tag{3.8}$$

Taking into account the equality

$$\begin{aligned} |\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy}| &= \Big(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy}\Big)^{1/2} \\ &= \Big(1 + (n^2 - 2n)(\boldsymbol{e}_{xy},\ \boldsymbol{e}_n)^2\Big)^{1/2}, \end{aligned}$$

by (3.8) we arrive at (3.3) for $p = 1$ with the sharp constant (3.4).

Let $1 < p \le \infty$. Using

$$\frac{1}{|y-x|^{nq}} = \frac{1}{x_n^{nq-n+1}} \left(\frac{x_n}{|y-x|}\right)^{n(q-1)} \frac{x_n}{|y-x|^n}, \quad \text{and} \quad \frac{x_n}{|y-x|} = (\boldsymbol{e}_{xy}, -\boldsymbol{e}_n),$$

and replacing q by $p/(p-1)$ in (3.7), we conclude that (3.3) holds with the sharp constant (3.5). □

Remark 3.1. Formula (3.5) for coefficient C_p can be written with the integral over the whole sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$,

$$C_p = \frac{2^{1/p}}{\omega_n} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}} \left|(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z})\right|^{p/(p-1)} \left|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\right|^{n/(p-1)} d\sigma \right\}^{(p-1)/p}.$$

Remark 3.2. For $p = \infty$ one can write (3.2) in the form

$$|\nabla u(x)| \le \frac{C_\infty}{x_n} \sup_{y \in \mathbb{R}^n_+} \left|u(y)\right|. \tag{3.9}$$

Hence,

$$|\nabla u(x)| \le \frac{C_\infty}{x_n} \sup_{y \in \mathbb{R}^n_+} \left|u(y) - \omega\right| \tag{3.10}$$

with an arbitrary constant ω. Minimizing (3.10) in ω, we obtain

$$|\nabla u(x)| \le \frac{C_\infty}{2x_n} \mathcal{O}_u\big(\mathbb{R}^n_+\big), \tag{3.11}$$

where $\mathcal{O}_u\big(\mathbb{R}^n_+\big)$ is the oscillation of u on $\mathbb{R}^n_+$. The last estimate is an analogue of (1.7) for harmonic functions in n-dimensional half-space.

The next assertion concerns the particular case $n=3$ in Proposition 3.1.

Theorem 3.1. *Let $u \in h^p(\mathbb{R}^3_+)$, and let x be an arbitrary point in $\mathbb{R}^3_+$. The sharp coefficient $\mathcal{C}_p(x)$ in the inequality*

$$|\nabla u(x)| \leq \mathcal{C}_p(x)\big\|u|_{x_3=0}\big\|_p \tag{3.12}$$

is given by

$$\mathcal{C}_p(x) = C_p x_3^{-(2+p)/p}\ , \tag{3.13}$$

where

$$C_1 = \frac{1}{\pi}\ , \tag{3.14}$$

and

$$C_p = \frac{1}{2^{1/p}\pi}\sup_{\gamma\geq 0}\ \frac{1}{\sqrt{1+\gamma^2}}\left\{\int_0^{\pi}\int_0^{\pi/2}\mathcal{F}_p(\varphi,\vartheta;\gamma)\ d\varphi d\vartheta\right\}^{(p-1)/p}, \tag{3.15}$$

if $1<p\leq\infty$. Here

$$\mathcal{F}_p(\varphi,\vartheta;\gamma) = \big|(3\cos^2\vartheta-1)+3\gamma\cos\vartheta\sin\vartheta\cos\varphi\big|^{p/(p-1)}\cos^{3/(p-1)}\vartheta\sin\vartheta. \tag{3.16}$$

In particular,

$$C_2 = \frac{1}{4}\sqrt{\frac{3}{\pi}}\ , \qquad C_\infty = \frac{4}{3\sqrt{3}}.$$

Proof. (i) Let $p=1$. The equality (3.14) follows from (3.4).

(ii) Let $1<p\leq\infty$. Since the integrand in (3.5) does not change when $\boldsymbol{z}$ is replaced by $-\boldsymbol{z}$, we may assume that $z_3=(-\boldsymbol{e}_3,\boldsymbol{z})\geq 0$. Introducing the notation $\boldsymbol{z}'=\boldsymbol{z}+z_3\boldsymbol{e}_3$, we see that $(\boldsymbol{z}',\boldsymbol{e}_3)=0$ and, hence $z_3^2+|\boldsymbol{z}'|^2=1$.

Let ϑ stand for the angle between the vectors $\boldsymbol{e}_\sigma$ and $-\boldsymbol{e}_3$. Since $\sigma\in\mathbb{S}^2_-$, we have $0\leq\vartheta\leq\pi/2$. Let $\boldsymbol{e}'_\sigma=\boldsymbol{e}_\sigma+\cos\vartheta\boldsymbol{e}_3$. Then $(\boldsymbol{e}'_\sigma,\boldsymbol{e}_3)=0, |\boldsymbol{e}'_\sigma|=\sin\vartheta$. We denote the angle between $\boldsymbol{e}'_\sigma$ and $\boldsymbol{z}'$ by φ with $0\leq\varphi\leq 2\pi$.

Let us find the expression for $\big(\boldsymbol{e}_3-3(\boldsymbol{e}_\sigma,\boldsymbol{e}_3)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)$ on $\mathbb{S}^2_-$ in spherical coordinates ϑ,φ:

$$\begin{aligned}
\big(\boldsymbol{e}_3-3(\boldsymbol{e}_\sigma,\boldsymbol{e}_3)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big) &= -z_3+3\cos\vartheta\big(\boldsymbol{e}_\sigma,\boldsymbol{z}\big)\\
&= -z_3+3\cos\vartheta\big(\boldsymbol{e}'_\sigma-\cos\vartheta\boldsymbol{e}_3,\ \boldsymbol{z}'-z_3\boldsymbol{e}_3\big)\\
&= -z_3+3\cos\vartheta\big[\big(\boldsymbol{e}'_\sigma,\boldsymbol{z}'\big)+z_3\cos\vartheta\big]\\
&= -z_3+3\cos\vartheta\big[|\boldsymbol{e}'_\sigma|\ |\boldsymbol{z}'|\cos\varphi+z_3\cos\vartheta\big]\\
&= \big(3\cos^2\vartheta-1\big)z_3+3|\boldsymbol{z}'|\cos\vartheta\sin\vartheta\cos\varphi.
\end{aligned}$$

This implies that the integral in (3.5) with $n=3$, that is

$$\int_{\mathbb{S}^2_-} F_{3,p}(\sigma,\boldsymbol{z})\ d\sigma,$$

can be written as

$$\int_0^{2\pi}\int_0^{\pi/2}\left|(3\cos^2\vartheta-1)z_3+3|\boldsymbol{z}'|\cos\vartheta\sin\vartheta\cos\varphi\right|^{p/(p-1)}\cos^{3/(p-1)}\vartheta\sin\vartheta\,d\varphi d\vartheta$$

$$=2\int_0^{\pi}\int_0^{\pi/2}\left|(3\cos^2\vartheta-1)z_3+3|\boldsymbol{z}'|\cos\vartheta\sin\vartheta\cos\varphi\right|^{p/(p-1)}\cos^{3/(p-1)}\vartheta\sin\vartheta\,d\varphi d\vartheta,$$

where we took into consideration that the integrand is 2π-periodic and even function of φ. Introducing the parameter $\gamma=|\boldsymbol{z}'|/z_3$ and using the equality $|\boldsymbol{z}'|^2+z_3^2=1$ together with (3.5) and (3.6), we arrive at (3.15).

(iii) Let $p=2$. By (3.15) and (3.16),

$$C_2=\frac{1}{\sqrt{2}\,\pi}\sup_{\gamma\geq 0}\frac{1}{\sqrt{1+\gamma^2}}\left\{\int_0^{\pi}\int_0^{\pi/2}\mathcal{F}_2(\varphi,\vartheta;\gamma)\,d\varphi d\vartheta\right\}^{1/2}, \tag{3.17}$$

where

$$\mathcal{F}_2(\varphi,\vartheta;\gamma)=\left[(3\cos^2\vartheta-1)+3\gamma\cos\vartheta\sin\vartheta\cos\varphi\right]^2\cos^3\vartheta\sin\vartheta. \tag{3.18}$$

The equalities (3.17) and (3.18) imply

$$C_2=\frac{1}{\sqrt{2}\pi}\sup_{\gamma\in\mathbb{R}_+}\frac{1}{\sqrt{1+\gamma^2}}\left\{\mathcal{I}_1+\gamma^2\mathcal{I}_2\right\}^{1/2}, \tag{3.19}$$

where

$$\mathcal{I}_1 = \int_0^{\pi}\int_0^{\pi/2}(3\cos^2\vartheta-1)^2\cos^3\vartheta\sin\vartheta\,d\varphi d\vartheta=\frac{3\pi}{8}, \tag{3.20}$$

$$\mathcal{I}_2 = 9\int_0^{\pi}\int_0^{\pi/2}\cos^5\vartheta\cos^2\varphi\sin^3\vartheta\,d\varphi d\vartheta=\frac{3\pi}{16}. \tag{3.21}$$

By (3.19) we have

$$C_2=\frac{1}{\sqrt{2}\pi}\max\left\{\mathcal{I}_1^{1/2},\mathcal{I}_2^{1/2}\right\},$$

which together with (3.20) and (3.21) gives the expression for C_2, indicated in the statement of the present assertion.

(iv) Let $p=\infty$. By (3.15) and (3.16),

$$C_\infty=\frac{1}{\pi}\sup_{\gamma\geq 0}\frac{1}{\sqrt{1+\gamma^2}}\int_0^{\pi}\int_0^{\pi/2}\mathcal{F}_\infty(\varphi,\vartheta;\gamma)\,d\varphi d\vartheta, \tag{3.22}$$

where

$$\mathcal{F}_\infty(\varphi,\vartheta;\gamma)=\left|(3\cos^2\vartheta-1)+3\gamma\cos\vartheta\sin\vartheta\cos\varphi\right|\sin\vartheta. \tag{3.23}$$

We are looking for a solution of the equation

$$(3\cos^2\vartheta-1)+3\gamma\cos\vartheta\sin\vartheta\cos\varphi=0 \tag{3.24}$$

as a function ϑ of φ. We can rewrite (3.24) as the second-order equation in $\tan\vartheta$:

$$\tan^2\vartheta-3\gamma\cos\varphi\tan\vartheta-2=0.$$

Since $0 \le \vartheta \le \pi/2$, we find that the nonnegative root of this equation is

$$\vartheta(\varphi) = \arctan\left(\frac{3\gamma\cos\varphi + \sqrt{8+9\gamma^2\cos^2\varphi}}{2}\right). \tag{3.25}$$

Taking into account that the left-hand side of (3.23) is nonnegative for $0 \le \vartheta \le \vartheta(\varphi)$, $0 \le \varphi \le \pi$, and using the equality

$$\int_0^{\pi}\int_0^{\pi/2} \big((3\cos^2\vartheta - 1) + 3\gamma\cos\varphi\cos\vartheta\sin\vartheta\big)\sin\vartheta d\varphi d\vartheta = 0,$$

we write (3.22) as

$$\begin{aligned} C_\infty &= \frac{2}{\pi}\sup_{\gamma\ge 0}\frac{1}{\sqrt{1+\gamma^2}}\int_0^{\pi}\int_0^{\vartheta(\varphi)}\big[(3\cos^2\vartheta-1)+3\gamma\cos\varphi\cos\vartheta\sin\vartheta\big]\sin\vartheta d\varphi d\vartheta \\ &= \frac{2}{\pi}\sup_{\gamma\ge 0}\frac{1}{\sqrt{1+\gamma^2}}\int_0^{\pi}\big[\cos\vartheta(\varphi)+\gamma\cos\varphi\sin\vartheta(\varphi)\big]\sin^2\vartheta(\varphi)d\varphi. \end{aligned} \tag{3.26}$$

By (3.25),

$$\begin{aligned} \sin\vartheta(\varphi) &= \frac{3\gamma\cos\varphi+\sqrt{8+9\gamma^2\cos^2\varphi}}{\sqrt{4+\left(3\gamma\cos\varphi+\sqrt{8+9\gamma^2\cos^2\varphi}\,\right)^2}}, \\ \cos\vartheta(\varphi) &= \frac{2}{\sqrt{4+\left(3\gamma\cos\varphi+\sqrt{8+9\gamma^2\cos^2\varphi}\,\right)^2}}. \end{aligned}$$

Therefore, the integrand in (3.26) can be written as

$$\begin{aligned} &\big[\cos\vartheta(\varphi)+\gamma\cos\varphi\sin\vartheta(\varphi)\big]\sin^2\vartheta(\varphi) \\ &= \frac{\left(2+\gamma\cos\varphi\left(3\gamma\cos\varphi+\sqrt{8+9\gamma^2\cos^2\varphi}\right)\right)\left(3\gamma\cos\varphi+\sqrt{8+9\gamma^2\cos^2\varphi}\right)^2}{\left(4+\left(3\gamma\cos\varphi+\sqrt{8+9\gamma^2\cos^2\varphi}\right)^2\right)^{3/2}} \\ &= \frac{\left(3\gamma\cos\varphi+\sqrt{8+9\gamma^2\cos^2\varphi}\right)^2}{6\left(4+\left(3\gamma\cos\varphi+\sqrt{8+9\gamma^2\cos^2\varphi}\right)^2\right)^{1/2}}. \end{aligned}$$

Hence, (3.26) becomes

$$C_\infty = \frac{1}{3\pi}\sup_{\gamma\ge 0}\frac{1}{\sqrt{1+\gamma^2}}\int_0^{\pi}\frac{\left(3\gamma\cos\varphi+\sqrt{8+9\gamma^2\cos^2\varphi}\right)^2}{\left(4+\left(3\gamma\cos\varphi+\sqrt{8+9\gamma^2\cos^2\varphi}\right)^2\right)^{1/2}}\,d\varphi. \tag{3.27}$$

Introducing the parameter

$$\alpha = \frac{3\gamma}{2\sqrt{2}}\ ,$$

we obtain

$$C_\infty = \frac{4}{\pi} \sup_{\alpha \geq 0} \frac{1}{\sqrt{9+8\alpha^2}} \int_0^\pi \frac{\left(\alpha\cos\varphi + \sqrt{1+\alpha^2\cos^2\varphi}\right)^2}{\left(1+2\left(\alpha\cos\varphi + \sqrt{1+\alpha^2\cos^2\varphi}\right)^2\right)^{1/2}}\, d\varphi. \quad (3.28)$$

The change of variable $t = \cos\varphi$ implies

$$C_\infty = \frac{4}{\pi} \sup_{\alpha \geq 0} \frac{1}{\sqrt{9+8\alpha^2}} \int_{-1}^1 \frac{\mathcal{P}(\alpha,t)}{\sqrt{1-t^2}}\, dt\ , \quad (3.29)$$

where

$$\mathcal{P}(\alpha,t) = \frac{\left(\alpha t + \sqrt{1+\alpha^2 t^2}\right)^2}{\left(1+2\left(\alpha t + \sqrt{1+\alpha^2 t^2}\right)^2\right)^{1/2}}. \quad (3.30)$$

Integrating in (3.29) over $(-1,0)$ and $(0,1)$ and using the equality $\mathcal{P}(\alpha,-t) = \mathcal{P}(-\alpha,t)$, we have

$$C_\infty = \frac{4}{\pi} \sup_{\alpha \geq 0} \frac{1}{\sqrt{9+8\alpha^2}} \int_0^1 \frac{\mathcal{P}(\alpha,t)+\mathcal{P}(-\alpha,t)}{\sqrt{1-t^2}}\, dt. \quad (3.31)$$

Applying the Schwarz inequality, we see that

$$C_\infty \leq \frac{4}{\pi} \sup_{\alpha \geq 0} \frac{1}{\sqrt{9+8\alpha^2}} \left\{ \int_0^1 \frac{\left(\mathcal{P}(\alpha,t)+\mathcal{P}(-\alpha,t)\right)^2}{\sqrt{1-t^2}}\, dt \right\}^{1/2} \left\{ \int_0^1 \frac{dt}{\sqrt{1-t^2}} \right\}^{1/2},$$

i.e.,

$$C_\infty \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \sup_{\alpha \geq 0} \frac{1}{\sqrt{9+8\alpha^2}} \left\{ \int_0^1 \frac{\left(\mathcal{P}(\alpha,t)+\mathcal{P}(-\alpha,t)\right)^2}{\sqrt{1-t^2}}\, dt \right\}^{1/2}. \quad (3.32)$$

By (3.30)

$$\left(\mathcal{P}(\alpha,t)+\mathcal{P}(-\alpha,t)\right)^2 = \frac{2\left(3+12\alpha^2 t^2 + 8\alpha^4 t^4 + \sqrt{9+8\alpha^2 t^2}\right)}{9+8\alpha^2 t^2}\ ,$$

hence, (3.32) takes the form

$$C_\infty \leq \frac{4}{\sqrt{\pi}} \sup_{\alpha \geq 0} \frac{1}{\sqrt{9+8\alpha^2}} \left\{ \int_0^1 \frac{3+12\alpha^2 t^2 + 8\alpha^4 t^4 + \sqrt{9+8\alpha^2 t^2}}{(9+8\alpha^2 t^2)\sqrt{1-t^2}}\, dt \right\}^{1/2}. \quad (3.33)$$

Since

$$\frac{3+12\alpha^2t^2+8\alpha^4t^4+\sqrt{9+8\alpha^2t^2}}{9+8\alpha^2t^2}=\frac{1}{\sqrt{9+8\alpha^2t^2}}+\frac{3+12\alpha^2t^2+8\alpha^4t^4}{9+8\alpha^2t^2}$$

$$\leq \frac{1}{3}+\alpha^2t^2+\frac{3}{8}-\frac{3}{8(9+8\alpha^2t^2)}=\frac{1}{3}+\alpha^2t^2+\frac{3}{8}-\frac{1}{24}+\left(\frac{1}{24}-\frac{3}{8(9+8\alpha^2t^2)}\right)$$

$$=\frac{2}{3}+\alpha^2t^2+\frac{\alpha^2t^2}{3(9+8\alpha^2t^2)}\leq\frac{2}{3}+\frac{28}{27}\alpha^2t^2\ ,$$

it follows from (3.33) that

$$\begin{aligned} C_\infty &\leq \frac{4}{\sqrt{\pi}}\sup_{\alpha\geq 0}\frac{1}{\sqrt{9+8\alpha^2}}\left\{\int_0^1\left(\frac{2}{3}+\frac{28}{27}\alpha^2t^2\right)\frac{dt}{\sqrt{1-t^2}}\right\}^{1/2} \\ &= \frac{4}{\sqrt{\pi}}\sup_{\alpha\geq 0}\frac{1}{\sqrt{9+8\alpha^2}}\left(\frac{\pi(9+7\alpha^2)}{27}\right)^{1/2}. \end{aligned} \tag{3.34}$$

Note that

$$\frac{d}{d\alpha}\left(\frac{9+7\alpha^2}{9+8\alpha^2}\right)=-\frac{18\alpha}{(9+8\alpha^2)^2}<0 \ \text{ for } \ \alpha>0,$$

therefore the supremum in α in the right-hand side of (3.34) is attained for $\alpha = 0$ and

$$C_\infty \leq \frac{4}{3\sqrt{3}}. \tag{3.35}$$

Besides, in view of (3.27),

$$C_\infty \geq \frac{1}{3\pi}\int_0^\pi \frac{8}{\sqrt{12}}\,d\varphi=\frac{4}{3\sqrt{3}},$$

which together with (3.35) proves the equality

$$C_\infty = \frac{4}{3\sqrt{3}},$$

given in the statement of the assertion. □

Remark 3.3. By Theorem 3.1 with $n = 3$, inequality (3.11) takes the form

$$|\nabla u(x)| \leq \frac{2}{3\sqrt{3}\,x_3}\ \mathcal{O}_u\big(\mathbb{R}^3_+\big)\,.$$

The sharp inequality is a three-dimensional analogue of the real-valued variant of (1.7).

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Cubature of Integral Operators by Approximate Quasi-interpolation

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To Vladimir Maz'ya on the occasion of his 70th birthday

Abstract. In this paper we report on some recent results concerning Hermite quasi-interpolation on uniform grids with interesting applications to the approximation of solutions to elliptic PDE, quasi-interpolation on nonuniform grids and the cubature of convolutions with radial kernel functions based on an approximation method proposed by V. Maz'ya.

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1. Introduction

In 1991 Maz'ya proposed a new approximation method called *approximate approximations*, mainly directed to the numerical solution of partial differential equations (see [7], [8]). The *approximate quasi-interpolant* on a uniform grid $\{h\,m\}$ has the simple form

$$\mathcal{M}_{h,\mathcal{D}}u(x) = \mathcal{D}^{-n/2} \sum_{m\in\mathbb{Z}^n} u(h\,m)\eta\left(\frac{x-h\,m}{h\sqrt{\mathcal{D}}}\right), \quad x \in \mathbb{R}^n \tag{1.1}$$

where h is the step size and $\mathcal{D}$ is a positive parameter; η is sufficiently smooth and of rapid decay in $\mathbb{R}^n$, and it is chosen such that the set of basis functions $\{\eta\left(\frac{x-m}{\sqrt{\mathcal{D}}}\right); m \in \mathbb{Z}^n\}$ represents merely an *approximate* partition of unity.

The main feature of the method is that, even if $\mathcal{M}_{h,\mathcal{D}}u(x)$ approximates $u(x)$ accurately, it does not converge to $u(x)$ as the grid size tends to zero. Namely, if $\mathcal{F}\eta - 1$ has a zero of order N at the origin ($\mathcal{F}\eta$ denotes the Fourier transform of η), then $\mathcal{M}_{h,\mathcal{D}}u(x)$ approximates $u(x)$ with order $\mathcal{O}(h^N)$ plus a non-convergent part called the *saturation error*. This lack of convergence is not perceptible in numerical computations because the saturation error can be chosen less than the precision machine or any prescribed accuracy with a proper choice of the parameter $\mathcal{D}$.

On the other hand this approximation process has considerable advantages due to the great flexibility in the choice of approximating functions η. This flexibility makes it possible to get simple but accurate formulas for the approximation of various integral and pseudo-differential operators. For example a cubature formula for the convolution integral

$$\mathcal{K}u(x) = \int_{\mathbb{R}^n} g(x-y)u(y)dy \tag{1.2}$$

can be defined by

$$\mathcal{K}\mathcal{M}_{h,\mathcal{D}}u(x) = \mathcal{D}^{-n/2} \sum_{m\in\mathbb{Z}^n} u(hm)\mathcal{K}\eta\left(\frac{\cdot - hm}{h\sqrt{\mathcal{D}}}\right)(x). \tag{1.3}$$

Therefore, it is sufficient to choose η such that $\mathcal{K}\eta$ can be computed efficiently. More, if $\mathcal{K}\eta$ is expressed analytically then (1.3) becomes a semi-analytic cubature formula for (1.2).

Some important applications have been considered in [10, 13, 16] (see also the book [15] and the review paper [17]) where formulas of various integral and pseudo-differential operators of mathematical physics have been obtained. Explicit semi-analytic time-marching algorithms for initial boundary value problems for linear and non linear evolution equations have been developed in [2, 9, 15].

In this paper we report on some recent results concerning Hermite quasi-interpolation on uniform grids with interesting applications to the approximation of solutions to elliptic PDE, quasi-interpolation on nonuniform grids and the cubature of convolutions with radial kernel functions based on the described quasi-interpolation processes.

2. Quasi-interpolation on uniform grids

Let us assume the following hypotheses concerning the function η:

I. η satisfies the decay condition.

$$|\eta(x)| \le c_0(1+|x|)^{-K},\ c_0>0,\ K>N+n$$

II. η satisfies the moment conditions

$$\int_{\mathbb{R}^n} x^\alpha \eta(x)dx = \delta_{\mathbf{0},\alpha}, \qquad 0 \le |\alpha| < N.$$

It is shown in [11] that, if $\{\partial^\alpha \mathcal{F}\eta(\sqrt{\mathcal{D}}\cdot)\} \in l_1(\mathbb{Z}^n)$, then $\mathcal{M}_{h,\mathcal{D}}u$ defined in (1.1) approximates a smooth function $u \in W_\infty^N(\mathbb{R}^n)$ with the following error estimates

$$|\mathcal{M}_{h,\mathcal{D}}u(x) - u(x)| \le c(h\sqrt{\mathcal{D}})^N ||\nabla_N u||_{L^\infty(\mathbb{R}^n)} + \sum_{|\alpha|=0}^{N-1} \left(\frac{h\sqrt{\mathcal{D}}}{2\pi}\right)^{|\alpha|} \frac{|\partial^\alpha u(x)|}{\alpha!} \sum_{\nu\in\mathbb{Z}^n\setminus\{0\}} |\partial^\alpha \mathcal{F}\eta(\sqrt{\mathcal{D}}\nu)|.$$

The constant c depends only on η and N.

If, additionally

III. η is differentiable up to the order of the smallest integer n_0 greater than $n/2$ and satisfies, together with its derivatives $\partial^\alpha\eta(x)$, $0 \le |\alpha| < n_0$, the decay condition **I**

then

$$\lim_{\mathcal{D}\to\infty} \sum_{\nu\in\mathbb{Z}^n\setminus\{0\}} |\partial^\alpha \mathcal{F}\eta(\sqrt{\mathcal{D}}\nu)| = 0.$$

Consequently, the simple formula (1.1) provides the following approximation result:

Theorem 2.1. ([14]) *If η is subjected to conditions* **I**, **II** *and* **III** *then for any $\epsilon > 0$ there exists sufficiently large $\mathcal{D}$ such that* (1.1) *approximates $u \in W^N_\infty(\mathbb{R}^n) \cap C^N(\mathbb{R}^n)$ with the error estimate*

$$|\mathcal{M}_{h,\mathcal{D}}u(x) - u(x)| \le c(h\sqrt{\mathcal{D}})^N ||\nabla_N u||_{L^\infty(\mathbb{R}^n)} + \epsilon \sum_{|\alpha|=0}^{N-1} (h\sqrt{\mathcal{D}})^{|\alpha|} |\partial^\alpha u(x)|. \quad (2.1)$$

The properties of (1.1) have been extensively studied in a series of papers ([7, 8, 10, 11, 12]) and in [15].

The approximation of a smooth function u, when the values of u and of some of its derivatives are prescribed at the nodes of a uniform grid, has been studied in [5]. More precisely, consider approximants of the form

$$\mathcal{M}_{h,\mathcal{D}}u(x) = \mathcal{D}^{-n/2} \sum_{m\in\mathbb{Z}^n} \eta\left(\frac{x-hm}{h\sqrt{\mathcal{D}}}\right) \mathcal{P}(-h\sqrt{\mathcal{D}}\partial)u(hm), \quad x \in \mathbb{R}^n \quad (2.2)$$

where the function $\mathcal{P}$ is a polynomial of degree at most $N-1$:

$$\mathcal{P}(t) = \sum_{|\alpha|=0}^{N-1} a_\alpha t^\alpha, \quad a_0 = 1, \qquad t \in \mathbb{R}^n.$$

Introduce the following extension of the moment conditions:

IV. η satisfies the conditions

$$\sum_{\alpha\le\beta} \frac{a_{\beta-\alpha}}{\alpha!} \int_{\mathbb{R}^n} x^\alpha \eta(x)dx = \delta_{\mathbf{0},\beta}, \quad 0 \le |\beta| < N.$$

Under hypotheses **IV** the approximants of the form (2.2) have the same approximation properties of (1.1), as expressed in the following

Theorem 2.2. ([5]) *Suppose that η satisfies the Conditions* **I**, **III** *and* **IV**. *Then for any $\epsilon > 0$ there exists $\mathcal{D} > 0$ such that, for all $u \in W_N^\infty(\mathbb{R}^n) \cap C^N(\mathbb{R}^n)$*

$$|\mathcal{M}_{h,\mathcal{D}}u(x) - u(x)| \leq \epsilon \sum_{|\beta|=0}^{N-1} (\sqrt{\mathcal{D}}h)^{|\beta|} |\partial^\beta u(x)| + c\,(\sqrt{\mathcal{D}}h)^N \, ||\nabla_N u||_{L^\infty(\mathbb{R}^n)}. \quad (2.3)$$

The constant c does not depend on h, u and $\mathcal{D}$.

We are interested in constructing generating functions satisfying condition **IV**. In fact any sufficiently smooth and rapidly decaying function ψ, with Fourier transform $\mathcal{F}\psi(0) \neq 0$, can be used to construct generating functions η satisfying the above-mentioned conditions.

If $\mathcal{A} = \{\mathcal{A}_{\alpha\beta}\}$ is the triangular matrix with the elements

$$\mathcal{A}_{\alpha\beta} = \begin{cases} a_{\beta-\alpha} & \alpha \leq \beta \\ 0 & \text{otherwise} \end{cases}, \ |\alpha|, |\beta| = 0, \dots, N-1,$$

and $\mathcal{A}^{-1} = \{\mathcal{A}^{(-1)}_{\alpha\beta}\}$ denotes its inverse, then the function

$$\eta(x) = \sum_{|\beta|=0}^{N-1} (-\partial)^\beta \psi(x) d_\beta; \quad d_\beta = \sum_{\alpha \leq \beta} \mathcal{A}^{(-1)}_{0\alpha} \frac{(-1)^{\beta-\alpha} \partial^{\beta-\alpha} (\mathcal{F}\psi)^{-1}(0)}{(\beta-\alpha)!(2\pi i)^{|\beta-\alpha|}} \quad (2.4)$$

satisfies condition **IV**.

For radial basis function $\psi(x) = \psi(|x|)$ formula (2.4) leads to

$$\eta(x) = \sum_{|\beta|=0}^{N-1} (-\partial)^\beta \psi(x) d_\beta; \quad d_\beta = \sum_{2\alpha \leq \beta} \mathcal{A}^{(-1)}_{0,\beta-2\alpha} \frac{(-1)^\alpha \partial^{2\alpha} (\mathcal{F}\psi)^{-1}(0)}{(2\alpha)!(2\pi)^{2|\alpha|}}. \quad (2.5)$$

If we apply formula (2.5) to the Gaussian $\psi(x) = \pi^{-n/2} \mathrm{e}^{-|x|^2}$, then we obtain the generating function

$$\eta(x) = \pi^{-n/2} \sum_{|\beta|=0}^{N-1} (-\partial)^\beta \, \mathrm{e}^{-|x|^2} \sum_{2\alpha \leq \beta} \frac{(-1)^\alpha}{\alpha! 4^{|\alpha|}} \mathcal{A}^{(-1)}_{0,\beta-2\alpha}. \quad (2.6)$$

In the case $N = 2M$ and $\mathcal{P}(x) \equiv 1$, (2.6) leads to the classical generating function $\eta(x) = \pi^{-n/2} L^{(n/2)}_{M-1}(|x|^2) \, \mathrm{e}^{-|x|^2}$ with the generalized Laguerre polynomials $L^{(n/2)}_{M-1}$ (see [12]).

Another interesting example is obtained by assuming $N = 2M$ and

$$\mathcal{P}(t) = \sum_{|\gamma|=0}^{M-1} \frac{(-1)^\gamma}{\gamma! 4^{|\gamma|}} t^{2\gamma}. \quad (2.7)$$

In this case the generating function satisfying condition **IV** is exactly the Gaussian $\eta(x) = \pi^{-n/2} \mathrm{e}^{-|x|^2}$ and (2.2) can be written in the form

$$\mathcal{M}_{h,\mathcal{D}}u(x) = (\pi\mathcal{D})^{-n/2} \sum_{m \in \mathbb{Z}^n} \sum_{j=0}^{M-1} (h\sqrt{\mathcal{D}})^{2j} \frac{(-1)^j}{j! 4^j} \Delta^j u(hm) \, \mathrm{e}^{-\frac{|x-hm|^2}{h^2 \mathcal{D}}}. \quad (2.8)$$

The quasi-interpolant (2.2) can be generalized to the case when the values of u are given on a lattice $\{h\,Cm\}$, with a real $n \times n$ non-singular matrix C. Define

$$\mathsf{M}_{h,\mathcal{D}}u(x) = \frac{\det C}{\mathcal{D}^{n/2}} \sum_{m\in\mathbb{Z}^n} \eta\left(\frac{x - h\,Cm}{h\sqrt{\mathcal{D}}}\right) \mathcal{P}(-h\sqrt{\mathcal{D}}\partial)u(h\,Cm), \quad x \in \mathbb{R}^n. \tag{2.9}$$

It can be easily seen that the quasi-interpolant $\mathsf{M}_{h,\mathcal{D}}$ provides the same approximation properties as $\mathcal{M}_{h,\mathcal{D}}$.

An application of formula (2.9) is the construction of quasi-interpolants on a regular triangular or hexagonal grid in the plane (see [4]).

The choice $N = 2M$ and $\psi(x) = \pi^{-n/2}\,\mathrm{e}^{-|x|^2}$ leads to the quasi-interpolants of order $2M$

$$\mathsf{M}_{h,\mathcal{D}}u(x) = \frac{\det C}{(\pi\mathcal{D})^{n/2}} \sum_{m\in\mathbb{Z}^n} u(h\,C\,m)\, L_{M-1}^{(n/2)}\left(\frac{|x - h\,Cm|^2}{h^2\mathcal{D}}\right) \mathrm{e}^{\frac{|x-h\,Cm|^2}{h^2\mathcal{D}}},$$

if $\mathcal{P} \equiv 1$, and

$$\mathsf{M}_{h,\mathcal{D}}u(x) = \frac{\det C}{(\pi\mathcal{D})^{n/2}} \sum_{m\in\mathbb{Z}^n} \sum_{j=0}^{M-1} \frac{(-h^2\mathcal{D})^j}{j!4^j} \Delta^j u(h\,C\,m)\, \mathrm{e}^{\frac{|x-h\,Cm|^2}{h^2\mathcal{D}}}, \tag{2.10}$$

if $\mathcal{P}$ is defined in (2.7).

Let $\mathcal{E}$ be the partial differential operator

$$\mathcal{E}u = \sum_{i,j}^{1,n} b_{ij}\partial_i\partial_j u = \langle B\nabla u, \nabla u \rangle, \quad B \in \mathbb{R}^{n\times n} \tag{2.11}$$

with a symmetric and positive definite matrix B. Choosing C such that $B^{-1} = C^T C$, it follows from the approximation properties of (2.10) that the approximate quasi-interpolant

$$M_{h,\mathcal{D}}u(x) = \frac{(\det B)^{-1/2}}{(\pi\mathcal{D})^{n/2}} \sum_{m\in\mathbb{Z}^n} \sum_{j=0}^{M-1} \frac{(-h^2\mathcal{D})^j}{j!4^j} \mathcal{E}^j u(hm)\, \mathrm{e}^{\langle B^{-1}(x-hm),x-hm\rangle/(h^2\mathcal{D})} \tag{2.12}$$

approximates $u(x)$ and the estimate (2.3) remains true.

2.1. Application to the solutions of elliptic PDEs

The quasi-interpolant (2.8) can be used for the approximation of solutions of harmonic equations. Suppose that u satisfies the equation

$$\Delta u(x) = 0, \qquad x \in \Omega. \tag{2.13}$$

If $\Omega = \mathbb{R}^n$ then $\mathcal{M}_{h,\mathcal{D}}$ has the simple form of the quasi-interpolation formula of second order

$$\mathcal{M}_{h,\mathcal{D}}u(x) = (\pi\mathcal{D})^{-n/2} \sum_{m\in\mathbb{Z}^n} u(hm)\, \mathrm{e}^{|x-hm|^2/(h^2\mathcal{D})} \tag{2.14}$$

but it provides higher approximation rates. More precisely

Theorem 2.3. ([5]) *If u is harmonic in $\mathbb{R}^n$ and, for any $y \in \mathbb{R}^n$, the series*

$$\sum_{|\beta|=0}^{\infty} \frac{\partial^\beta u(x)}{\sqrt{\beta!}} y^\beta$$

is absolutely convergent, then, for $\sqrt{\mathcal{D}}h < 1$, the quasi-interpolant (2.14) *approximates u with*

$$\mathcal{M}_{h,\mathcal{D}} u(x) - u(x) = \sum_{m \in \mathbb{Z}^n \setminus \{0\}} \tilde{u}(x + \pi i h \mathcal{D} m) \, \mathrm{e}^{-\pi^2 \mathcal{D} |\nu|^2} \, \mathrm{e}^{2\pi i \langle \frac{x}{h}, \nu \rangle} .$$

Here $\tilde{u}$ denotes the analytic extension of $u(\cdot)$ onto $\mathbb{C}^n$.

A similar effect of higher than second-order approximation for the simple quasi-interpolation can also be seen for solutions of the equation $\mathcal{E}u(x) = 0$ with the PDE (2.11) in a domain $\Omega \subset \mathbb{R}^n$. For those functions, extended to zero outside Ω, the quasi-interpolant (2.12) takes the form of the second-order quasi-interpolant

$$M_{h,\mathcal{D}} u(x) = \frac{(\det B)^{-1/2}}{(\pi \mathcal{D})^{n/2}} \sum_{hm \in \Omega} u(hm) \, \mathrm{e}^{\langle B^{-1}(x-hm), x-hm \rangle / (h^2 \mathcal{D})} , \tag{2.15}$$

but approximates u with an improved rate.

Theorem 2.4. ([5]) *Suppose that $\mathcal{E}u = 0$ in a convex domain $\Omega \subset \mathbb{R}^n$ and, for a given $N = 2M$,*

$$C_u = \sum_{|\beta|=2M} \|\partial^\beta u\|_{L_\infty(\Omega)} \prod_{\beta_j > 0} \sqrt{\frac{2}{(\beta_j - 1)!}} < \infty .$$

Then for any $\epsilon > 0$ and for any subdomain $\Omega' \subsetneq \Omega$ there exist $\mathcal{D} > 0$ and $h > 0$ such that the quasi-interpolant (2.15) *provides for all $x \in \Omega'$ the estimate*

$$|M_{h,\mathcal{D}} u(x) - u(x)| \leq C_u (h\sqrt{\mathcal{D}})^N c + \epsilon \sum_{|\beta|=0}^{N-1} (h\sqrt{\mathcal{D}})^{|\beta|} |\partial^\beta u(x)| , \tag{2.16}$$

where the constant c depends only on the space dimension.

3. Quasi-interpolation on more general grids

The extension of approximate quasi-interpolation when the set of nodes is a smooth image of a uniform grid has been studied in [14]. It was shown that formulas similar to (1.1) maintain the basic properties of approximate quasi-interpolation. The case of piecewise uniform grid has been considered in [1].

3.1. Quasi-interpolation with perturbed uniform grid

In [4] formula (1.1) is modified to the case of functions given on a set of scattered nodes close to a uniform grid. Precisely,

A. Let X_h be a sequence of grids with the property that there exist a uniform grid Λ and a positive κ, not depending on h, such that for any $y_m \in \Lambda$, the ball $B(hy_m, h\kappa) \cap X_h \neq \emptyset$.
Let us suppose that the quasi-interpolant on the uniform grid $h\,\Lambda$:

$$\mathsf{M}_{h,\mathcal{D}} u(x) = \mathcal{D}^{-n/2} \sum_{y_m \in \Lambda} u(h\, y_m) \eta \left(\frac{x - h\, y_m}{h\sqrt{\mathcal{D}}} \right), \quad x \in \mathbb{R}^n$$

approximates any smooth function u with the error estimate (2.1).

B. Denote by $\tilde{x}_m$ the node of the grid X_h closest to hy_m and by $\mathrm{st}(\tilde{x}_m) \subset X_h$ a collection of $m_N = (N-1+n)!/((n!(N-1)! - 1)$ nodes x_k such that the Vandermonde matrix

$$V_{h,m} = \left\{ \left(\frac{x_k - \tilde{x}_m}{h} \right)^{\alpha} \right\}, \ |\alpha| = 1, \dots, N-1, k = 1, \dots, m_N$$

is not singular.
There exists $\kappa_1 > 0$ such that $\mathrm{st}(\tilde{x}_m) \subset B(\tilde{x}_m, h\kappa_1)$ with $|V_{h,m}| \geq c > 0$ uniformly in h.

If $V_{h,m}^{-1} = \{b_{\alpha,k}^{(m)}\}$ is the inverse matrix of $V_{h,m}$, define the functional

$$\mathbb{F}_{h,m}(u) = u(\widetilde{x}_m) \Bigg(1 - \sum_{|\alpha|=1}^{N-1} \left(y_m - \frac{\widetilde{x}_m}{h} \right)^{\alpha} \sum_{x_k \in \mathrm{st}(\widetilde{x}_m)} b_{\alpha,k}^{(m)} \Bigg)$$
$$+ \sum_{x_k \in \mathrm{st}(\widetilde{x}_m)} u(x_k) \sum_{|\alpha|=1}^{N-1} b_{\alpha,k}^{(m)} \left(y_m - \frac{\widetilde{x}_m}{h} \right)^{\alpha}.$$

Theorem 3.1. ([4]) *Under the conditions* **A** *and* **B**, *for any* $\epsilon > 0$ *there exists* $\mathcal{D} > 0$ *such that the quasi-interpolant*

$$\mathbb{M}_{h,\mathcal{D}} u(x) = \mathcal{D}^{-n/2} \sum_{y_m \in \Lambda} \mathbb{F}_{h,m}(u) \eta \left(\frac{x - hy_m}{h\sqrt{\mathcal{D}}} \right)$$

approximates any $u \in W_N^\infty(\mathbb{R}^n) \cap C^N(\mathbb{R}^n)$ *with*

$$|\mathbb{M}_{h,\mathcal{D}} u(x) - u(x)| \leq c\, h^N \|\nabla_N u\|_{L_\infty(\mathbb{R}^n)} + \epsilon \sum_{|\beta|=0}^{N-1} (h\sqrt{\mathcal{D}})^{|\beta|} |\partial^\beta u(x)|\,,$$

where c *does not depend on* u *and* h.

In (3.1) the generating functions are centered at the nodes of the uniform grid $\{hy_m\}$. This can be helpful to design fast methods for the approximation of the convolution integrals (1.2). The value $\mathcal{K}u(h\,y_k)$ is approximated by

$$\mathcal{K}\mathbb{M}_{h,\mathcal{D}} u(h\, y_k) = h^n \sum_{y_m \in \Lambda} \mathbb{F}_{h,m}(u) a_{k-m}^{(h)}$$

with the coefficients

$$a_{k-m}^{(h)} = \int_{\mathbb{R}^n} g(h(y_k - y_m - \sqrt{\mathcal{D}}))\eta(y)\,dy.$$

3.2. Quasi-interpolation with scattered grids

By modifying the approximating functions, the method of approximate quasi-interpolation has been generalized to functions with values given on more general grids (see [3], [15]). In particular one can achieve the approximation of a smooth function u with arbitrary order N up to a small saturation error as long as an approximate partition of unity, centered at the scattered nodes, does exist.

Under some mild restrictions on the scattered nodes, an approximate partition of unit can be obtained from a given system of rapidly decaying approximating functions if these functions are multiplied by polynomials.

Conditions on the grid $X = \{x_m\}$ and on the scaling parameters h_m, which ensure the existence of polynomials $\mathcal{P}_m$ of degree uniformly bounded such that the function

$$\Theta(x) = \sum_{x_m \in X} \mathcal{P}_m\left(\frac{x - x_m}{h_m}\right)\eta\left(\frac{x - x_m}{h_m}\right) \tag{3.1}$$

approximates 1, are given in [3] and in [15]. The function η is supposed to be sufficiently smooth, of rapid decay and

$$\sum_{x_m \in X} \eta\left(\frac{x - x_m}{h_m}\right) \geq c > 0, \qquad \forall x \in \mathbb{R}^n.$$

For a fixed $\epsilon > 0$, let us assume the existence of polynomials $\mathcal{P}_m$ such that Θ in (3.1) satisfies the estimate

$$|\Theta(x) - 1| < \epsilon, \qquad \forall x \in \mathbb{R}^n.$$

Using the function Θ one can construct quasi-interpolants of high order of approximation rate up to the saturation error, without additional requirements on η, as the moment conditions.

For $x_m \in X$, denote by $\mathrm{st}(x_m)$ a collection of m_N nodes in X such that the Vandermonde matrix

$$\{(x_k - x_m)^\alpha\}, \quad |\alpha| = 1, \ldots, N-1,\ x_k \in \mathrm{st}(x_m)\,,$$

is not singular. The union of the node x_m and its star $\mathrm{st}(x_m)$ is denoted by $\bar{\mathrm{st}}(x_m)$.

Let us assume the following hypothesis concerning the grid:

C. For any $x_m \in X$ there exists a ball $B(x_m, h_m)$ which contains m_N nodes $x_k \in \mathrm{st}(x_m)$ such that

$$|V_{h_m,m}| = \left|\det\left\{\left(\frac{x_k - x_m}{h_m}\right)^\alpha\right\}\right| \geq c > 0$$

with c not depending on x_m.

Denote by $\{b^{(m)}_{\alpha,k}\}$ the elements of the inverse of $V_{h_m,m}$ and define the polynomials

$$\mathcal{Q}_{m,k}(y) = \sum_{|\alpha|=1}^{N-1} b^{(m)}_{\alpha,k} y^\alpha \, \mathcal{P}_m(y) \,, \qquad \text{if} \qquad x_k \in \mathrm{st}(x_m)\,;$$

$$\mathcal{Q}_{m,m}(y) = \left(1 - \sum_{x_k \in \mathrm{st}(x_m)} \sum_{|\alpha|=1}^{N-1} b^{(m)}_{\alpha,k} y^\alpha \right) \mathcal{P}_m(y) \,.$$

Theorem 3.2. ([3]) *Let $\epsilon > 0$ arbitrary. The quasi-interpolant*

$$Mu(x) = \sum_{x_k \in X} u(x_k) \sum_{\substack{m:\\ \bar{\mathrm{st}}(x_m) \ni x_k}} \mathcal{Q}_{m,k}\left(\frac{x - x_m}{h_m}\right) \eta \left(\frac{x - x_m}{h_m}\right) \tag{3.2}$$

approximates $u \in W^N_\infty(\mathbb{R}^n)$ with the error estimate

$$|Mu(x) - u(x)| \le C \sup_m h_m^N ||\nabla_N u||_{L_\infty(\mathbb{R}^n)} + \epsilon |u(x)|.$$

The constant C does not depend on u and ϵ.

For the special case of scattered nodes close to a piecewise uniform grid, a method to construct polynomials $\mathcal{P}_m$ such that the sum

$$\sum_{x_m \in X} \mathcal{P}_m(x) \mathrm{e}^{-|x - x_m|^2/h_m^2}$$

approximates the function 1 with any given precision, is described in [3] and, more in detail, in [15].

The starting point is that, for a given bounded domain $\Omega \subset \mathbb{R}^n$, there exist a finite sequence of nodes $G = \{g_k\}$ belonging to a piecewise uniform grid, parameters $\mathcal{D}_k$ and numbers a_k such that

$$\left\{ a_k \mathrm{e}^{-|x - g_k|^2/\mathcal{D}_k} \,:\, g_k \in G \right\} \tag{3.3}$$

forms an approximate partition of unity (see [15]). For $\epsilon > 0$, let us assume that the approximate partition of unity (3.3) satisfies the condition

$$\left| 1 - \sum_{g_k \in G} a_k \mathrm{e}^{-|x - g_k|^2/\mathcal{D}_k} \right| < \epsilon, \ \forall x \in \Omega. \tag{3.4}$$

Suppose that a finite set of scattered nodes $X = \{x_m\}$ satisfies the following condition:

D. For any $\epsilon > 0$, there exist a piecewise uniform grid G and an approximate partition of unity (3.3) which satisfies (3.4). For some fixed $\kappa > 1$ and for each $g_k \in G$ there exists a subset of scattered nodes $\Sigma(g_k) \subset X$ such that

$$\text{if} \ \ x_i \in \Sigma(g_k) \ \ \text{then} \ \ x_i \in B(g_k, \kappa\sqrt{\mathcal{D}_k}); \qquad \cup_{g_k \in G} \Sigma(g_k) = X.$$

The method consists in the approximation of each Gaussian $\mathrm{e}^{-|x-g_k|^2/\mathcal{D}_k}$ in (3.4) with functions of the form polynomial times Gaussians centered at the scattered nodes $x_i \in \Sigma(g_k)$. Hence one has to determine a polynomial $\mathcal{P}_i^{(k)}$ such that

$$\sum_{x_i \in \Sigma(g_k)} \mathcal{P}_i^{(k)}(x)\mathrm{e}^{-|x-x_i|^2/h_i^2} \approx \mathrm{e}^{-|x-g_k|^2/\mathcal{D}_k}.$$

A least square method for constructing $\mathcal{P}_i^{(k)}$ is used and the approximation is very accurate provided that the degree of the polynomial is large enough. The choice of the scaling parameters h_i is decisive and has been analyzed in detail in [15].

Therefore the function

$$\sum_{g_k \in G} a_k \sum_{x_i \in \Sigma(g_k)} \mathcal{P}_i^{(k)}(x)\mathrm{e}^{-|x-x_i|^2/h_i^2}$$

is the required approximate partition of unity.

The proposed method for constructing an approximate partition of unity has the advantage that it does not require to solve a large algebraic system. Instead, in order to obtain locally an analytic representation of the partition of unity and consequently of the quasi-interpolant (3.2), one has to solve a small number of linear systems of moderate size.

4. Application to the computation of integral operators

A direct application of formula (3.2) for $\eta(x) = \mathrm{e}^{-|x|^2}$ is the computation of integral operators with radial kernel

$$\mathcal{K}u(x) = \int_{\mathbb{R}^n} g(|x-y|)\, u(y)\, dy\,, \tag{4.1}$$

to which many important high-dimensional integral operators in mathematical physics are closely related.

If the density u is replaced by its quasi-interpolant

$$Mu(x) = \sum_{x_k \in X} u(x_k) \sum_{\substack{m:\\ \bar{\mathrm{st}}(x_m)\ni x_k}} \mathcal{Q}_{m,k}\left(\frac{x-x_m}{h_m}\right)\mathrm{e}^{-|x-x_m|^2/h_m^2},$$

then one obtains the cubature formula

$$\mathcal{K}Mu(x) = \sum_{x_k \in X} u(x_k) \sum_{\substack{m:\\ \bar{\mathrm{st}}(x_m)\ni x_k}} h_m^n\, \mathcal{T}_{m,k}(h_m\,\partial_x)\, \mathcal{L}_m\left(\frac{x-x_m}{h_m}\right) \tag{4.2}$$

for the integral (4.1). Here $\mathcal{T}_{m,k}(x)$ are polynomials defined by the relation

$$\mathcal{Q}_{m,k}(x)\mathrm{e}^{-|x|^2} = \mathcal{T}_{m,k}(\partial_x)\mathrm{e}^{-|x|^2}$$

and $\mathcal{L}_m$ is the action of the integral operator on the Gaussian

$$\mathcal{L}_m(x) = \int_{\mathbb{R}^n} g(h_m|x-y|)\,\mathrm{e}^{-|y|^2}\, dy\,.$$

For the applicability of the cubature formula (4.2) efficient methods for computing of the values of

$$\mathcal{T}_{m,k}(h_m\,\partial_x)\,\mathcal{L}_m\left(\frac{x-x_m}{h_m}\right) \tag{4.3}$$

are necessary. The integral $\mathcal{L}_m$ can be transformed to a one-dimensional integral by different ways:

1. Using the Fourier transform of g

$$\mathcal{F}g(t) = \frac{2\pi}{t^{\,n/2-1}} \int_0^\infty g(\tau)\,J_{n/2-1}(2\pi r\tau)\,\tau^{\,n/2}\,d\tau\,,$$

one gets

$$\mathcal{L}_m(x) = \frac{2}{h_m^n|x|^{n/2-1}} \int_0^\infty \rho^{\,n/2}\ \mathrm{e}^{-\rho^2}\ \mathcal{F}g\Big(\frac{\rho}{\pi h_m}\Big)\,J_{n/2-1}(2\rho|x|)\ d\rho\,, \tag{4.4}$$

where J_n is the Bessel function of the first kind.

2. The convolution formula for radial functions with the Gaussian (see [15]) leads to

$$\mathcal{L}_m(x) = \frac{2\,\pi^{n/2}\ \mathrm{e}^{-|x|^2}}{|x|^{n/2-1}} \int_0^\infty \rho^{\,n/2}\,\mathrm{e}^{-\rho^2}\,g(h_m\rho)\,I_{n/2-1}(2\rho|x|)\,d\rho \tag{4.5}$$

with the modified Bessel functions of the first kind I_n.

In many cases the integrals (4.4) or (4.5) can be expressed by some special functions and therefore also the terms (4.3) in the cubature formula. Examples of those analytic formulas are given in [15]. However, from the numerical point of view it is sometimes more advantageous to use a quadrature method for one-dimensional integrals with smooth and sufficiently elementary integrands instead of computing the values of special functions.

As example we list some volume potentials of elliptic partial differential operators which admit one-dimensional integral representations by elementary functions.

3. By solving certain related parabolic or hyperbolic equations the following formulas have been derived in [15]:

- The operator $-\Delta + a^2$, $a \in \mathbb{R}$, has the fundamental solution

$$g(x) = \frac{1}{(2\pi)^{n/2}}\Big(\frac{a}{|x|}\Big)^{n/2-1} K_{n/2-1}(a|x|)\,,$$

  where K_n is the modified Bessel function of the second kind. The volume potential of the Gaussian can be expressed as

$$\mathcal{L}_m(x) = \frac{1}{4h_m^{n-2}} \int_0^\infty \frac{\mathrm{e}^{-a^2h_m^2\rho/4}\ \mathrm{e}^{\,-|x|^2/(1+\rho)}}{(1+\rho)^{n/2}}\,d\rho\,. \tag{4.6}$$

- The fundamental solution of the Helmholtz operator $-\Delta - k^2$, $k \in \mathbb{R}$, is

$$g(x) = \frac{i}{4}\Big(\frac{k}{2\pi|x|}\Big)^{n/2-1} H^{(1)}_{n/2-1}(k|x|)\,,$$

where $H_n^{(1)} = J_n + iY_n$ is the Hankel function of the first kind. Then

$$\mathcal{L}_m(x) = \frac{i}{4h_m^{n-2}} \int_0^\infty \frac{\mathrm{e}^{ik^2 h_m^2 \rho/4}\, \mathrm{e}^{-|x|^2/(1+i\,\rho)}}{(1+i\,\rho)^{n/2}}\, d\rho\,.$$

- The fundamental solution of the three-dimensional Lamé operator

$$-\mu\,\Delta - (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}$$

is the Kelvin-Somigliana matrix $\|\Gamma_{kl}\|_{3\times 3}$ with

$$\Gamma_{kl}(x) = \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)} \left(\frac{\lambda+3\mu}{\lambda+\mu} \frac{\delta_{kl}}{|x|} + \frac{x^k\, x^l}{|x|^3} \right),$$

where we denote $x = (x^1, x^2, x^3) \in \mathbb{R}^3$. The action on the Gaussian is given by the integral

$$(\mathcal{L}_m)_{kl}(x) = \int_{\mathbb{R}^3} \Gamma_{kl}\big(h_m(x-y)\big)\; \mathrm{e}^{-|y|^2}\; dy \tag{4.7}$$
$$= \frac{1}{4\mu h_m} \int_0^\infty \left(\delta_{kl}\Big(1 + \frac{(\lambda+3\mu)\rho}{2(\lambda+2\mu)}\Big) + \frac{x^k x^l (\lambda+\mu)\rho}{(\lambda+2\mu)(1+\rho)} \right) \frac{\mathrm{e}^{-|x|^2/(1+\rho)}}{(1+\rho)^{5/2}}\, d\rho\,.$$

In any of these examples the computation of

$$\mathcal{T}_{m,k}(h_m\,\partial_x)\,\mathcal{L}_m\left(\frac{x-x_m}{h_m}\right)$$

requires again the quadrature of one-dimensional integrals with smooth and sufficiently elementary integrands. One has to replace

$$\mathrm{e}^{-|x-x_m|^2/(h_m^2(1+\rho))} \quad \text{by} \quad \mathcal{T}_{m,k}((1+\rho)^{-1/2}\partial_y)\, \mathrm{e}^{-|y|^2}\Big|_{y=\frac{x-x_m}{h_m\sqrt{1+\rho}}}\,.$$

Correspondingly, for the elastic potential the term

$$\frac{(x^k - x_m^k)(x^l - x_m^l)}{h_m^2(1+\rho)}\, \mathrm{e}^{-|x-x_m|^2/(h_m^2(1+\rho))}$$

has to be replaced by the differential expression

$$\mathcal{T}_{m,k}((1+\rho)^{-1/2}\partial_y)\, y^k y^l\; \mathrm{e}^{-|y|^2}\Big|_{y=\frac{x-x_m}{h_m\sqrt{1+\rho}}}\,.$$

For the quadrature of the resulting one-dimensional integrals we recommend their transformation to integrals over $\mathbb{R}$ with doubly exponentially decaying integrands and to use the classical trapezoidal rule. It is well known that it is exponentially converging for certain classes of integrands, for example periodic functions and rapidly decaying functions on the real line. For example, Poisson's summation formula yields that

$$\tau \sum_{k=-\infty}^{\infty} f(k\tau) = \sum_{j=-\infty}^{\infty} \mathcal{F}f\big(\frac{2\pi j}{\tau}\big)$$

for any sufficiently smooth function, say of the Schwarz class $\mathcal{S}(\mathbb{R})$.

It follows from

$$\Big|\int_{-\infty}^{\infty} f(t)\,dt - \tau \sum_{k=-K}^{K} f(k\tau)\Big| \le \Big|\sum_{j\neq 0} \mathcal{F}f\big(\frac{2\pi j}{\tau}\big)\Big| + \tau\Big|\sum_{|k|>K} f(k\tau)\Big|$$

that if f and $\mathcal{F}f$ both decay very rapidly, the trapezoidal rule can provide accurate approximations for moderate values of the step size τ and the number of terms $2K+1$.

We recommend the procedure proposed by J. Waldvogel [18] to make the integrand doubly exponentially decaying, i.e., $|f(t)| \le c\exp(-\alpha\exp(|t|))$ for $|t| \to \infty$ with certain constants $c, \alpha > 0$.

By the substitution $\rho = \mathrm{e}^{\xi}$ the integration domain is transformed to $\mathbb{R}$. Then the required decay can be obtained by the (repeated) substitutions $\xi = u + \mathrm{e}^{u}$ and $u = v - \mathrm{e}^{-v}$.

Numerical tests show that due to the fast decay of f this approach is very efficient for integrals of the type (4.6) and (4.7). The trapezoidal rules require only a small amount of terms in the quadrature sum, and they are nearly independent on the differential operators $\mathcal{T}_{m,k}$. More details also concerning tensor product approximation of the action of the integrals operators on Gaussian are given in [6].

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Pointwise Estimates for the Polyharmonic Green Function in General Domains

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Abstract. In the present paper we establish sharp estimates on the polyharmonic Green function and its derivatives in an arbitrary bounded open set.

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1. Introduction

Let Ω be a arbitrary bounded domain in $\mathbb{R}^n$. The Green function for the polyharmonic equation is a function $G : \Omega \times \Omega \to \mathbb{R}$ which for every fixed $y \in \Omega$ solves the equation

$$(-\Delta_x)^m G(x,y) = \delta(x-y), \qquad x \in \Omega, \tag{1.1}$$

in the space $\mathring{W}^{m,2}(\Omega)$, a completion of $C_0^\infty(\Omega)$ in the norm given by $\|u\|_{\mathring{W}^{m,2}(\Omega)} = \|\nabla^m u\|_{L^2(\Omega)}$. The case $m = 2$ corresponds to the biharmonic equation, and respectively, (1.1) gives rise to the biharmonic Green function.

In dimension two the biharmonic Green function can be interpreted as a deflection of a thin clamped plate under a point load. Numerous applications in structural engineering, emerging from this fact, have stimulated considerable interest to the biharmonic equation and its Green function as early as in the beginning of 20th century. In 1908 Hadamard has published a volume devoted to properties of the solutions to the biharmonic equation [8], where, in particular, he conjectured that the corresponding Green function must be positive, at least, in convex domains. However, several counterexamples to Hadamard's conjecture have been found later on ([5], [6], [7], [12], [20], [3], [9]) and it was proved that the biharmonic Green function may change sign even in a smooth convex domain, in a sufficiently eccentric ellipse ([7], [3]). Moreover, in a rectangle the first eigenfunction of the

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biharmonic operator has infinitely many changes of sign near each of the vertices ([2], [9]).

During the past century, the biharmonic and more generally, the polyharmonic Green function has been thoroughly studied, and a variety of upper estimates has been obtained. In particular, we would like to point out the results in smooth domains [4], [11], [18], [19], in conical domains [16], [10], and in polyhedra [17].

The objective of the present paper is to establish sharp estimates on the polyharmonic Green function and its derivatives without any geometric assumptions, in an arbitrary bounded open set.

For example, we show that, whenever the dimension $n \in [3, 2m+1] \cap \mathbb{N}$ is odd, the regular part of the Green function admits the estimate

$$|\nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \nabla_y^{m-\frac{n}{2}+\frac{1}{2}} (G(x,y) - \Gamma(x-y))| \leq \frac{C}{\max\{d(x), d(y), |x-y|\}}, \quad x, y \in \Omega, \tag{1.2}$$

where $\Gamma(x) = C_{m,n}|x|^{2m-n}$, $x \in \Omega$, is a fundamental solution for the polyharmonic operator, $d(x)$ is the distance from $x \in \Omega$ to $\partial\Omega$ and the constant C depends on n and m only. Hence, in particular,

$$|\nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \nabla_y^{m-\frac{n}{2}+\frac{1}{2}} G(x,y)| \leq \frac{C}{|x-y|}, \qquad x, y \in \Omega, \tag{1.3}$$

and similar results are established for the lower-order derivatives.

Furthermore, the estimates on the Green function allow us to derive optimal bounds for the solution u of the Dirichlet boundary value problem

$$(-\Delta)^m u = \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} c_\alpha \partial^\alpha f_\alpha, \quad u \in \mathring{W}^{m,2}(\Omega), \tag{1.4}$$

where c_α are some constants. Specifically,

$$|\nabla^{m-\frac{n}{2}+\frac{1}{2}} u(x)| \leq C \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} \int_\Omega d(y)^{m-\frac{n}{2}+\frac{1}{2}-|\alpha|} \frac{|f_\alpha(y)|}{|x-y|}\, dy, \; x \in \Omega, \tag{1.5}$$

whenever the integrals on the right-hand side of (1.5) are finite. In particular, there exists a constant $C_\Omega > 0$ depending on m, n and the domain Ω such that

$$\|\nabla^{m-\frac{n}{2}+\frac{1}{2}} u\|_{L^\infty(\Omega)} \quad \leq \quad C_\Omega \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} \|d(\cdot)^{m-\frac{n}{2}-\frac{1}{2}-|\alpha|} f_\alpha\|_{L^p(\Omega)}, \tag{1.6}$$

for $p > \frac{n}{n-1}$.

The bounds above are sharp, in the sense that the solution of the polyharmonic equation in an arbitrary domain generally does not exhibit more regularity. Indeed, assume that $n \in [3, 2m+1] \cap \mathbb{N}$ is odd and let $\Omega \subset \mathbb{R}^n$ be the punctured unit ball $B_1 \setminus \{O\}$, where $B_r = \{x \in \mathbb{R}^n : |x| < r\}$. Consider a function $\eta \in C_0^\infty(B_{1/2})$ such that $\eta = 1$ on $B_{1/4}$. Then let

$$u(x) := \eta(x)\, \partial_x^{m-\frac{n}{2}-\frac{1}{2}} \Gamma(x) = C\eta(x)\, \partial_x^{m-\frac{n}{2}-\frac{1}{2}} (|x|^{2m-n}), \qquad x \in B_1 \setminus \{O\}, \tag{1.7}$$

where ∂_x stands for a derivative in the direction of x_i for some $i = 1, \dots, n$. It is straightforward to check that $u \in \mathring{W}^{m,2}(\Omega)$ and $(-\Delta)^m u \in C_0^\infty(\Omega)$. While $\nabla^{m-\frac{n}{2}+\frac{1}{2}} u$ is bounded, the derivatives of the order $m - \frac{n}{2} + \frac{3}{2}$ are not, and moreover, $\nabla^{m-\frac{n}{2}+\frac{1}{2}} u$ is not continuous at the origin. Therefore, the estimates (1.5) are optimal in general domains.

We also derive full analogues of (1.2), (1.3), (1.5), (1.6) and accompanying lower-order bounds in even dimensions. In that case, the optimal regularity turns out to be of the order $m - \frac{n}{2}$.

Finally, we would like to mention that the Green function estimates in this paper generalize the earlier developments in [13], where the biharmonic Green function was treated, and [15], where the pointwise estimates on polyharmonic Green function have been established in dimensions $2m + 1$ and $2m + 2$ for $m > 2$ and dimensions $5, 6, 7$ for $m = 2$.

2. Preliminaries

The Green function estimates in the present paper are based, in particular, on the recent results for locally polyharmonic functions that will appear in [14]. We record them below without the proof. Here and throughout the paper $B_r(Q)$ and $S_r(Q)$ denote, respectively, the ball and the sphere with radius r centered at Q and $C_{r,R}(Q) = B_R(Q) \setminus \overline{B_r(Q)}$. When the center is at the origin, we write B_r in place of $B_r(O)$, and similarly $S_r := S_r(O)$ and $C_{r,R} := C_{r,R}(O)$. Also, $\nabla^m u$ stands for a vector of all derivatives of u of the order m.

Proposition 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $2 \le n \le 2m+1$, $Q \in \mathbb{R}^n \setminus \Omega$, and $R > 0$. Suppose*

$$(-\Delta)^m u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}(Q)), \quad u \in \mathring{W}^{m,2}(\Omega). \tag{2.1}$$

Then

$$\frac{1}{\rho^{2\lambda+n-1}} \int_{S_\rho(Q)\cap\Omega} |u(x)|^2 \, d\sigma_x \le \frac{C}{R^{2\lambda+n}} \int_{C_{R,4R}(Q)\cap\Omega} |u(x)|^2 \, dx \quad \text{for every } \rho < R, \tag{2.2}$$

where C is a constant depending on m and n only, and

$$\lambda = m - n/2 + 1/2 \text{ when } n \text{ is odd}, \qquad \lambda = m - n/2 \text{ when } n \text{ is even}. \tag{2.3}$$

Moreover, for every $x \in B_{R/4}(Q) \cap \Omega$

$$|\nabla^i u(x)|^2 \le C \frac{|x - Q|^{2\lambda - 2i}}{R^{n+2\lambda}} \int_{C_{R/4,4R}(Q)\cap\Omega} |u(y)|^2 \, dy, \qquad 0 \le i \le \lambda, \tag{2.4}$$

where λ is given by (2.3).

In addition, using the Kelvin transform, estimates near the origin for solutions of elliptic equations can be translated into estimates at infinity. In particular, Proposition 2.1 leads to the following result (also proved in [14]).

Proposition 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $2 \le n \le 2m+1$, $Q \in \mathbb{R}^n \setminus \Omega$, $r > 0$ and assume that*

$$(-\Delta)^m u = f \text{ in } \Omega, \quad f \in C_0^\infty(B_{r/4}(Q) \cap \Omega), \quad u \in \mathring{W}^{m,2}(\Omega). \tag{2.5}$$

Then

$$\rho^{2\lambda+n+1-4m} \int_{S_\rho(Q)\cap\Omega} |u(x)|^2 \, d\sigma_x \le C\, r^{2\lambda+n-4m} \int_{C_{r/4,r}(Q)\cap\Omega} |u(x)|^2 \, dx, \tag{2.6}$$

for any $\rho > r$ and λ given by (2.3).

Furthermore, for any $x \in \Omega \setminus B_{4r}(Q)$

$$|\nabla^i u(x)|^2 \le C \, \frac{r^{2\lambda+n-4m}}{|x-Q|^{2\lambda+2n-4m+2i}} \int_{C_{r/4,4r}(Q)\cap\Omega} |u(y)|^2 \, dy, \qquad 0 \le i \le \lambda. \tag{2.7}$$

3. Estimates for the Green function

Following [1] we point out that the fundamental solution for the m-Laplacian is a linear combination of the characteristic singular solution (defined below) and any m-harmonic function in $\mathbb{R}^n$. The characteristic singular solution is

$$C_{m,n}|x|^{2m-n}, \qquad \text{if } n \text{ is odd, or if } n \text{ is even with } n \ge 2m+2, \tag{3.1}$$

$$C_{m,n}|x|^{2m-n}\log|x|, \qquad \text{if } n \text{ is even with } n \le 2m. \tag{3.2}$$

The exact expressions for constants $C_{m,n}$ can be found in [1], p. 8. For the purposes of this paper we will use the fundamental solution given by

$$\Gamma(x) = C_{m,n} \begin{cases} |x|^{2m-n}, & \text{if } n \text{ is odd,} \\ |x|^{2m-n}\log\frac{\operatorname{diam}\Omega}{|x|}, & \text{if } n \text{ is even and } n \le 2m, \\ |x|^{2m-n}, & \text{if } n \text{ is even and } n \ge 2m+2. \end{cases} \tag{3.3}$$

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then there exist constants C, C' depending on m and n only such that for every $x, y \in \Omega$ the following estimates hold. If $n \in [3, 2m+1] \cap \mathbb{N}$ is odd then*

$$|\nabla_x^i \nabla_y^j G(x,y)| \le C \min\left\{1, \left(\frac{d(x)}{|x-y|}\right)^{m-\frac{n}{2}+\frac12-i}, \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}+\frac12-j}\right\} \times \frac{1}{|x-y|^{n-2m+i+j}}, \tag{3.4}$$

whenever $0 \le i, j \le m - \frac{n}{2} + \frac{1}{2}$ *are such that* $i + j \ge 2m - n$, *and*

$$\begin{aligned} |\nabla_x^i \nabla_y^j G(x,y)| &\le C \min\left\{1, \left(\frac{d(x)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-i}, \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-j}\right\} \\ &\quad \times \frac{1}{|x-y|^{n-2m+i+j}} \min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\}^{n-2m+i+j} \end{aligned} \tag{3.5}$$

if $0 \le i, j \le m - \frac{n}{2} + \frac{1}{2}$ *are such that* $i + j \le 2m - n$.

If $n \in [2, 2m] \cap \mathbb{N}$ *is even, then*

$$\begin{aligned} |\nabla_x^i \nabla_y^j G(x,y)| &\le C \min\left\{1, \left(\frac{d(x)}{|x-y|}\right)^{m-\frac{n}{2}-i}, \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}-j}\right\} \\ &\quad \times \frac{1}{|x-y|^{n-2m+i+j}} \min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\}^{n-2m+i+j} \\ &\quad \times \log\left(1 + \frac{\min\{d(x), d(y)\}}{|x-y|}\right), \end{aligned} \tag{3.6}$$

for all $0 \le i, j \le m - \frac{n}{2}$.

Furthermore, the estimates on the regular part of the Green function $S(x,y) = G(x,y) - \Gamma(x-y)$, $x, y \in \Omega$, *are as follows. If* $n \in [3, 2m+1] \cap \mathbb{N}$ *is odd then*

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}}, \tag{3.7}$$

whenever $0 \le i, j \le m - \frac{n}{2} + \frac{1}{2}$ *are such that* $i + j \ge 2m - n$, *and*

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{|x-y|^{n-2m+i+j}} \min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\}^{n-2m+i+j}, \tag{3.8}$$

if $0 \le i, j \le m - \frac{n}{2} + \frac{1}{2}$ *are such that* $i + j \le 2m - n$.

If $n \in [2, 2m] \cap \mathbb{N}$ *is even, then*

$$\begin{aligned} |\nabla_x^i \nabla_y^j S(x,y)| &\le \frac{C}{|x-y|^{n-2m+i+j}} \min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\}^{n-2m+i+j} \\ &\quad \times \log\left(1 + \frac{\operatorname{diam}\Omega}{\max\{d(x), d(y), |x-y|\}}\right), \end{aligned} \tag{3.9}$$

for all $0 \le i, j \le m - \frac{n}{2}$.

Proof. Let us start with some auxiliary calculations. Let α be a multi-index of length less than or equal to λ, where λ is given by (2.3). Then $\partial_y^\alpha \Gamma(x-y)$ can be written as

$$\partial_y^\alpha \Gamma(x-y) = P^\alpha(x-y) \log \frac{\operatorname{diam}\Omega}{|x-y|} + Q^\alpha(x-y). \tag{3.10}$$

When the dimension is odd, $P^\alpha \equiv 0$. If the dimension is even (and less than or equal to $2m$ by the assumptions of the theorem) then P^α is a homogeneous polynomial of order $2m-n-|\alpha|$ as long as $|\alpha| \le 2m-n$. In any case, Q^α is a homogeneous function of order $2m-n-|\alpha|$.

Consider a function η such that

$$\eta \in C_0^\infty(B_{1/2}) \quad \text{and} \quad \eta = 1 \quad \text{in} \quad B_{1/4}, \tag{3.11}$$

and define

$$\mathcal{R}_\alpha(x,y) := \partial_y^\alpha G(x,y) - \eta\left(\frac{x-y}{d(y)}\right)\left(P^\alpha(x-y)\log\frac{d(y)}{|x-y|} + Q^\alpha(x-y)\right), \tag{3.12}$$

for $x, y \in \Omega$. Also, let us denote

$$\begin{aligned} f_\alpha(x,y) &:= (-\Delta_x)^m \mathcal{R}_\alpha(x,y) \\ &= -\left[(-\Delta_x)^m, \eta\left(\frac{x-y}{d(y)}\right)\right]\left(P^\alpha(x-y)\log\frac{d(y)}{|x-y|} + Q^\alpha(x-y)\right). \end{aligned} \tag{3.13}$$

It is not hard to see that for every α as above

$$f_\alpha(\cdot,y) \in C_0^\infty(C_{d(y)/4,d(y)/2}(y)) \qquad \text{and} \qquad |f_\alpha(x,y)| \le C d(y)^{-n-|\alpha|}, \quad x,y \in \Omega. \tag{3.14}$$

Then for every fixed $y \in \Omega$ the function $x \mapsto \mathcal{R}_\alpha(x,y)$ is a solution of the boundary value problem

$$(-\Delta_x)^m \mathcal{R}_\alpha(x,y) = f_\alpha(x,y) \text{ in } \Omega, \quad f_\alpha(\cdot,y) \in C_0^\infty(\Omega), \quad \mathcal{R}_\alpha(\cdot,y) \in \mathring{W}^{m,2}(\Omega), \tag{3.15}$$

so that

$$\|\nabla_x^m \mathcal{R}_\alpha(\cdot,y)\|_{L^2(\Omega)} = \|\mathcal{R}_\alpha(\cdot,y)\|_{W^{m,2}(\Omega)} \le C\|f_\alpha(\cdot,y)\|_{W^{-m,2}(\Omega)}, \qquad 0 \le |\alpha| \le \lambda. \tag{3.16}$$

Here $W^{-m,2}(\Omega)$ stands for the Banach space dual of $\mathring{W}^{m,2}(\Omega)$, i.e.,

$$\|f_\alpha(\cdot,y)\|_{W^{-m,2}(\Omega)} = \sup_{v \in \mathring{W}^{m,2}(\Omega):\, \|v\|_{\mathring{W}^{m,2}(\Omega)}=1} \int_\Omega f_\alpha(x,y) v(x)\, dx. \tag{3.17}$$

Recall that by Hardy's inequality

$$\left\|\frac{v}{|\cdot - Q|^m}\right\|_{L^2(\Omega)} \le C\,\|\nabla^m v\|_{L^2(\Omega)} \quad \text{for every} \quad v \in \mathring{W}^{m,2}(\Omega), \quad Q \in \partial\Omega. \tag{3.18}$$

Then for some $y_0 \in \partial\Omega$ such that $|y-y_0| = d(y)$ and any v in (3.17)

$$\begin{aligned} \int_\Omega f_\alpha(x,y)v(x)\,dx &\le C\left\|\frac{v}{|\cdot - y_0|^m}\right\|_{L^2(\Omega)} \|f_\alpha(\cdot,y)|\cdot - y_0|^m\|_{L^2(\Omega)} \\ &\le C d(y)^m \,\|\nabla^m v\|_{L^2(\Omega)}\, \|f_\alpha(\cdot,y)\|_{L^2(C_{d(y)/4,d(y)/2}(y))}, \end{aligned} \tag{3.19}$$

and therefore, by (3.14)

$$\|\nabla_x^m \mathcal{R}_\alpha(\cdot,y)\|_{L^2(\Omega)} \le C d(y)^{m-|\alpha|-n/2}. \tag{3.20}$$

Now we split the discussion into a few cases.

Case I: $|x-y| \geq Nd(y)$ **or** $|x-y| \geq Nd(x)$ for some large N to be specified later.

Let us first assume that $|x-y| \geq Nd(y)$. As before, we denote by y_0 some point on the boundary such that $|y-y_0| = d(y)$. Then by (3.14)–(3.15) the function $x \mapsto \mathcal{R}_\alpha(x,y)$ is m-harmonic in $\Omega \setminus B_{3d(y)/2}(y_0)$. Hence, by Proposition 2.2 with $r = 6d(y)$

$$|\nabla_x^i \mathcal{R}_\alpha(x,y)|^2 \leq C\,\frac{d(y)^{2\lambda+n-4m}}{|x-y_0|^{2\lambda+2n-4m+2i}} \int_{C_{3d(y)/2,24d(y)}(y_0)} |\mathcal{R}_\alpha(z,y)|^2\,dz, \quad (3.21)$$

provided that $0 \leq i \leq \lambda$ and $|x-y| \geq 4r + d(y)$, i.e., $N \geq 25$. The right-hand side of (3.21) is bounded by

$$\begin{aligned} &C\,\frac{d(y)^{2\lambda+n-2m}}{|x-y_0|^{2\lambda+2n-4m+2i}} \int_{C_{3d(y)/2,24d(y)}(y_0)} \frac{|\mathcal{R}_\alpha(z,y)|^2}{|z-y_0|^{2m}}\,dz \\ &\leq C\,\frac{d(y)^{2\lambda+n-2m}}{|x-y_0|^{2\lambda+2n-4m+2i}} \int_\Omega |\nabla_z^m \mathcal{R}_\alpha(z,y)|^2\,dz \\ &\leq C\,\frac{d(y)^{2\lambda-2|\alpha|}}{|x-y|^{2\lambda+2n-4m+2i}}, \end{aligned} \quad (3.22)$$

by Hardy's inequality and (3.20). Therefore,

$$|\nabla_x^i \mathcal{R}_\alpha(x,y)|^2 \leq C\,\frac{d(y)^{2\lambda-2|\alpha|}}{|x-y|^{2\lambda+2n-4m+2i}}, \quad \text{when} \quad |x-y| \geq Nd(y), \quad 0 \leq i, |\alpha| \leq \lambda. \quad (3.23)$$

Since for $N \geq 25$ the condition $|x-y| \geq Nd(y)$ guarantees that $\eta\left(\frac{x-y}{d(y)}\right) = 0$ and hence, $\mathcal{R}_\alpha(x,y) = \partial_y^\alpha G(x,y)$ when $|x-y| \geq Nd(y)$, the estimate (3.23) with $j := |\alpha|$ implies

$$|\nabla_x^i \nabla_y^j G(x,y)|^2 \leq C\,\frac{d(y)^{2\lambda-2j}}{|x-y|^{2\lambda+2n-4m+2i}}, \quad \text{when} \quad |x-y| \geq Nd(y), \quad 0 \leq i,j \leq \lambda. \quad (3.24)$$

Also, by the symmetry of the Green function we automatically deduce that

$$|\nabla_x^i \nabla_y^j G(x,y)|^2 \leq C\,\frac{d(x)^{2\lambda-2i}}{|x-y|^{2\lambda+2n-4m+2j}}, \quad \text{when} \quad |x-y| \geq Nd(x), \quad 0 \leq i,j \leq \lambda. \quad (3.25)$$

In particular, (3.24) and (3.25) combined give the estimate

$$|\nabla_x^i \nabla_y^j G(x,y)| \leq \frac{C}{|x-y|^{n-2m+i+j}}, \quad \text{when} \quad |x-y| \geq N\min\{d(x), d(y)\}, \quad (3.26)$$

for $0 \leq i,j \leq \lambda$.

Now further consider several cases. If n is odd, then

$$|\nabla_x^i \nabla_y^j \Gamma(x-y)| \leq \frac{C}{|x-y|^{n-2m+i+j}} \qquad \text{for all} \qquad x,y \in \Omega, \quad i,j \geq 0, \quad (3.27)$$

while if n is even, then

$$|\nabla_x^i \nabla_y^j \Gamma(x-y)| \leq C_1 \, |x-y|^{-n+2m-i-j} \log \frac{\operatorname{diam}(\Omega)}{|x-y|} + C_2 \, |x-y|^{-n+2m-i-j}, \quad (3.28)$$

for all $x, y \in \Omega$ and $0 \leq i + j \leq 2m - n$.

Combining this with (3.26) we deduce that for $n \leq 2m + 1$ odd

$$|\nabla_x^i \nabla_y^j S(x,y)| \leq \frac{C}{|x-y|^{n-2m+i+j}} \quad \text{when} \quad |x-y| \geq N \min\{d(x), d(y)\}, \quad (3.29)$$

while if n is even, then

$$|\nabla_x^i \nabla_y^j S(x,y)| \leq C \, |x-y|^{-n+2m-i-j} \left(C' + \log \frac{\operatorname{diam}(\Omega)}{|x-y|} \right) \quad (3.30)$$

provided that $|x - y| \geq N \min\{d(x), d(y)\}$ and $0 \leq i, j \leq \lambda$.

Case II: $|x - y| \leq N^{-1} d(y)$ **or** $|x - y| \leq N^{-1} d(x)$.

Assume that $|x-y| \leq N^{-1} d(y)$. For such x we have $\eta\left(\frac{x-y}{d(y)}\right) = 1$ and therefore

$$\mathcal{R}_\alpha(x,y) = \partial_y^\alpha \, G(x,y) - P^\alpha(x-y) \log \frac{d(y)}{|x-y|} - Q^\alpha(x-y). \quad (3.31)$$

Hence, if n is odd,

$$\mathcal{R}_\alpha(x,y) = \partial_y^\alpha \left(G(x,y) - \Gamma(x-y) \right), \quad \text{when} \quad |x-y| \leq N^{-1} d(y), \quad (3.32)$$

and if n is even,

$$\mathcal{R}_\alpha(x,y) = \partial_y^\alpha \left(G(x,y) - \Gamma(x-y) \right) + P^\alpha(x-y) \log \frac{\operatorname{diam} \Omega}{d(y)}, \quad (3.33)$$

when $|x-y| \leq N^{-1} d(y)$. By the interior estimates for solutions of elliptic equations

$$|\nabla_x^i \mathcal{R}_\alpha(x,y)|^2 \leq \frac{C}{d(y)^{n+2i}} \int_{B_{d(y)/8}(x)} |\mathcal{R}_\alpha(z,y)|^2 \, dz, \quad \text{for any } i \leq m, \quad (3.34)$$

since the function $\mathcal{R}_\alpha$ is m-harmonic in $B_{d(y)/4}(y) \supset B_{d(y)/8}(x)$. Now we bound the expression above by

$$\frac{C}{d(y)^{n+2i-2m}} \int_{B_{d(y)/4}(y)} \frac{|\mathcal{R}_\alpha(z,y)|^2}{|z-y_0|^{2m}} \, dz \leq \frac{C}{d(y)^{n+2i-2m}} \, \|\nabla_x^m \mathcal{R}(\cdot, y)\|_{L^2(\Omega)}^2$$
$$\leq \frac{C}{d(y)^{2n-4m+2i+2|\alpha|}}, \quad (3.35)$$

with $0 \leq |\alpha| \leq \lambda$.

Let us now focus on the case of n odd. It follows from (3.32) and (3.34)–(3.35) that

$$|\nabla_x^i \nabla_y^j S(x,y)| \leq \frac{C}{d(y)^{n-2m+i+j}}, \quad 0 \leq i \leq m, \quad 0 \leq j \leq \lambda, \quad |x-y| \leq N^{-1} d(y), \quad (3.36)$$

and hence, by symmetry,

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{d(x)^{n-2m+i+j}}, \quad 0 \le i \le \lambda, \quad 0 \le j \le m, \quad |x-y| \le N^{-1} d(x). \tag{3.37}$$

However, we have

$$|x-y| \le N^{-1} d(y) \quad \Longrightarrow \quad (N-1)\, d(y) \le N d(x) \le (N+1)\, d(y), \tag{3.38}$$

i.e., $d(y) \approx d(x)$ whenever $|x-y|$ is less than or equal to either $N^{-1}d(y)$ or $N^{-1}d(x)$. Therefore, when the dimension is odd,

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{\max\{d(x), d(y)\}^{n-2m+i+j}}, \tag{3.39}$$

provided that $|x-y| \le N^{-1} \max\{d(x), d(y)\}$, $0 \le i,j \le \lambda$, $i+j \ge 2m-n$, and

$$|\nabla_x^i \nabla_y^j S(x,y)| \le C \min\{d(x), d(y)\}^{2m-n-i-j}, \tag{3.40}$$

for $|x-y| \le N^{-1} \max\{d(x), d(y)\}$, $0 \le i,j \le \lambda$, $i+j \le 2m-n$.

As for the Green function itself, we then have for $|x-y| \le N^{-1} \max\{d(x), d(y)\}$

$$|\nabla_x^i \nabla_y^j G(x,y)| \le \frac{C}{|x-y|^{n-2m+i+j}}, \quad \text{if } i+j \ge 2m-n, \tag{3.41}$$

and

$$|\nabla_x^i \nabla_y^j G(x,y)| \le C \min\{d(x), d(y)\}^{2m-n-i-j}, \quad \text{if } i+j \le 2m-n, \tag{3.42}$$

with i,j such that $0 \le i,j \le \lambda$.

Similar considerations apply to the case when the dimension is even, leading to the following results:

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{d(y)^{n-2m+i+j}} \left(C' + \log \frac{\operatorname{diam}\Omega}{d(y)} \right), \tag{3.43}$$

for $0 \le i \le m$, $0 \le j \le \lambda$, $|x-y| \le N^{-1} d(y)$, and

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{d(x)^{n-2m+i+j}} \left(C' + \log \frac{\operatorname{diam}\Omega}{d(x)} \right), \tag{3.44}$$

for $0 \le i \le \lambda$, $0 \le j \le m$, $|x-y| \le N^{-1} d(x)$. In particular, in view of (3.38), and the fact that $2m-n-i-j \ge 0$ whenever $0 \le i,j \le \lambda$ and n is even, we have

$$|\nabla_x^i \nabla_y^j S(x,y)| \le C \min\{d(x), d(y)\}^{2m-n-i-j} \left(C' + \log \frac{\operatorname{diam}\Omega}{\max\{d(x), d(y)\}} \right), \tag{3.45}$$

for $|x-y| \le N^{-1} \max\{d(x), d(y)\}$, $0 \le i,j \le \lambda$.

Passing to the Green function estimates, (3.31) and (3.34)–(3.35) lead to the bound

$$|\nabla_x^i \nabla_y^j G(x,y)| \le C\, d(y)^{2m-n-i-j} \left(C' + \log \frac{d(y)}{|x-y|} \right), \tag{3.46}$$

for $0 \le i,j \le \lambda$, $|x-y| \le N^{-1}d(y)$. Hence, by symmetry,

$$|\nabla_x^i \nabla_y^j G(x,y)| \le C\, d(x)^{2m-n-i-j} \left(C' + \log \frac{d(x)}{|x-y|}\right), \tag{3.47}$$

for $0 \le i,j \le \lambda$, $|x-y| \le N^{-1}d(x)$, and therefore,

$$|\nabla_x^i \nabla_y^j G(x,y)| \le C \min\{d(x),d(y)\}^{2m-n-i-j} \left(C' + \log \frac{\min\{d(x),d(y)\}}{|x-y|}\right), \tag{3.48}$$

for all $0 \le i,j \le \lambda$, and $|x-y| \le N^{-1}\max\{d(x),d(y)\}$.

Finally, it remains to consider

Case III: $|x-y| \approx d(y) \approx d(x)$, or more precisely, the situation when

$$N^{-1}\, d(x) \le |x-y| \le N d(x) \quad \text{and} \quad N^{-1}\, d(y) \le |x-y| \le N d(y). \tag{3.49}$$

In this case we use the m-harmonicity of $x \mapsto G(x,y)$ in $B_{d(x)/(2N)}(x)$. Let $x_0 \in \partial\Omega$ be such that $|x-x_0| = d(x)$. By the interior estimates,

$$\begin{aligned}
|\nabla_x^i \nabla_y^{|\alpha|} G(x,y)|^2 &\le \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} G(z,y)|^2\, dz \\
&\le \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} \Gamma(z-y)|^2\, dz \\
&\qquad + \frac{C}{d(x)^{n+2i-2m}} \int_{B_{2d(x)}(x_0)} \frac{|\mathcal{R}_\alpha(z,y)|^2}{|z-x_0|^{2m}}\, dz \\
&\le \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} \Gamma(z-y)|^2\, dz \\
&\qquad + \frac{C}{d(x)^{n+2i-2m}} \int_{\Omega} |\nabla_z^m \mathcal{R}_\alpha(z,y)|^2\, dz \\
&\le \frac{C}{d(x)^{2n-4m+2i+2|\alpha|}} + \frac{C}{d(x)^{n-2m+2i} d(y)^{n-2m+2|\alpha|}},
\end{aligned} \tag{3.50}$$

provided that $0 \le i \le m$, $0 \le |\alpha| \le \lambda$ and n is odd. The right-hand side of (3.50) also provided the estimate on derivatives of the Green function holds when n is even, upon observing that

$$\begin{aligned}
\frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} G(z,y)|^2\, dz &\le \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |P^\alpha(z-y)|^2\, dz \\
&+ \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |Q^\alpha(z-y)|^2\, dz \\
&+ \frac{C}{d(x)^{n+2i-2m}} \int_{B_{2d(x)}(x_0)} \frac{|\mathcal{R}_\alpha(z,y)|^2}{|z-x_0|^{2m}}\, dz,
\end{aligned} \tag{3.51}$$

since the absolute value of $\log \frac{|z-y|}{d(y)}$ is bounded by a constant for z, x, y as in (3.51), (3.49).

Hence, for x, y satisfying (3.49) we have

$$|\nabla_x^i \nabla_y^j G(x,y)| \leq \frac{C}{\min\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \approx \frac{C}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}}, \quad (3.52)$$

for $0 \leq i, j \leq \lambda$.

When n is odd, the same argument implies the following estimate on a regular part of Green function

$$|\nabla_x^i \nabla_y^j S(x,y)| \leq \frac{C}{\min\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \approx \frac{C}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}}, \quad (3.53)$$

for $0 \leq i, j \leq \lambda$, and x, y satisfying (3.49). If n is even, however, we are led to a bound

$$\begin{aligned} |\nabla_x^i \nabla_y^j S(x,y)| &\leq \frac{C}{\min\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \\ &\quad \times \left(C' + \log \frac{\operatorname{diam}\Omega}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}}\right) \\ &\approx \frac{C}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \\ &\quad \times \left(C' + \log \frac{\operatorname{diam}\Omega}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}}\right) \quad (3.54) \end{aligned}$$

for $0 \leq i, j \leq \lambda$.

The final bounds for the Green function are a combination of estimates (3.24), (3.25), (3.41), (3.42), (3.48), (3.52). It helps to observe that the regions of $(x, y) \in \Omega \times \Omega$ in (3.24), (3.25) are disjoint from those in (3.41), (3.42), (3.48). The condition $|x-y| \leq N^{-1} \max\{d(x), d(y)\}$ excludes the possibility of $|x-y| \leq N^{-1} \max\{d(x), d(y)\}$. This is, in particular, due to (3.38). Also, the bound (3.52) is the same as (3.24), (3.25), (3.41), (3.42) for the case when $d(x)$, $d(y)$ and $|x-y|$ are all comparable. Hence, it can be suitably absorbed. Finally, it is straightforward to check that

$$C' + \log \frac{\min\{d(x), d(y)\}}{|x-y|} \approx \log\left(1 + \frac{\min\{d(x), d(y)\}}{|x-y|}\right) \quad (3.55)$$

for $|x-y| \leq N^{-1} \max\{d(x), d(y)\}$.

Analogously, the desired estimates on the regular part of the Green function can be drawn from (3.29), (3.30), (3.39), (3.40). (3.45), (3.53), (3.54). □

4. Applications: estimates on solutions of the Dirichlet problem

Green function estimates proved in Section 3 allow us to investigate the solutions of the Dirichlet problem for the polyharmonic equation for a wide class of data.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $n \in [3, 2m+1] \cap \mathbb{N}$ is odd. Consider the boundary value problem*

$$(-\Delta)^m u = \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} c_\alpha \partial^\alpha f_\alpha \in W^{-m,2}(\Omega), \quad u \in \mathring{W}^{m,2}(\Omega). \tag{4.1}$$

Then the solution satisfies the estimate

$$|\nabla^{m-\frac{n}{2}+\frac{1}{2}} u(x)| \leq C \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} \int_\Omega d(y)^{m-\frac{n}{2}+\frac{1}{2}-|\alpha|} \frac{|f_\alpha(y)|}{|x-y|}\, dy, \; x \in \Omega, \tag{4.2}$$

whenever the integrals on the right-hand side of (4.2) *are finite. The constant C in* (4.2) *depends on m and n only.*

In particular, there exists a constant $C_\Omega > 0$ depending on m, n and the domain Ω such that

$$\|\nabla^{m-\frac{n}{2}+\frac{1}{2}} u\|_{L^\infty(\Omega)} \quad \leq \quad C_\Omega \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} \|d(\cdot)^{m-\frac{n}{2}-\frac{1}{2}-|\alpha|} f_\alpha\|_{L^p(\Omega)}, \tag{4.3}$$

for $p > \frac{n}{n-1}$, provided that the norms on the right-hand side of (4.3) *are finite.*

Proof. Indeed, the integral representation formula

$$u(x) = \int_\Omega G(x,y) \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} c_\alpha \partial^\alpha f_\alpha(y)\, dy, \qquad x \in \Omega, \tag{4.4}$$

follows directly from the definition of Green function. It implies that

$$\nabla^{m-\frac{n}{2}+\frac{1}{2}} u(x) = \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} c_\alpha (-1)^{|\alpha|} \int_\Omega \nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \partial_y^\alpha G(x,y) f_\alpha(y)\, dy. \tag{4.5}$$

Furthermore, due to the estimate (3.4) with $i = j = m - \frac{n}{2} + \frac{1}{2}$ we have

$$\int_\Omega \left| \nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \nabla_y^{m-\frac{n}{2}+\frac{1}{2}} G(x,y) \right| |f(y)|\, dy \leq C \int_\Omega \frac{|f(y)|}{|x-y|}\, dy, \tag{4.6}$$

while the bounds in (3.5) can be used to show that for every $j \leq m - \frac{n}{2} - \frac{1}{2}$

$$\int_\Omega \left| \nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \nabla_y^j G(x,y) \right| |f(y)|\, dy \leq C \int_\Omega \min \left\{ 1, \left(\frac{d(y)}{|x-y|} \right)^{m-\frac{n}{2}+\frac{1}{2}-j} \right\}$$

$$\times \frac{1}{|x-y|^{\frac{n}{2}-m+\frac{1}{2}+j}} \min \left\{ \frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1 \right\}^{\frac{n}{2}-m+\frac{1}{2}+j} |f(y)|\, dy. \tag{4.7}$$

We split the latter integral to the cases $|x-y| \le N^{-1}d(y)$ and $|x-y| \ge N^{-1}d(y)$ with $N \ge 25$ (as in Theorem 3.1). Recall that according to (3.38) in the first case $d(x) \approx d(y)$ and therefore

$$\min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\} \approx \frac{|x-y|}{d(y)} \quad \text{when} \quad |x-y| \le N^{-1}d(y), \tag{4.8}$$

while in the second case $d(x) \le |x-y| + d(y) \le (1+N)|x-y|$, so that

$$\min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\} \approx C \quad \text{when} \quad |x-y| \ge N^{-1}d(y). \tag{4.9}$$

Hence, the expression on the right-hand side of (4.7) can be further estimated by

$$\begin{aligned}
&C\int_{y\in\Omega:\,|x-y|\le N^{-1}d(y)} \frac{1}{|x-y|^{\frac{n}{2}-m+\frac{1}{2}+j}} \left(\frac{|x-y|}{d(y)}\right)^{\frac{n}{2}-m+\frac{1}{2}+j} |f(y)|\,dy \\
&\qquad + C\int_{y\in\Omega:\,|x-y|\ge N^{-1}d(y)} \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-j} \frac{1}{|x-y|^{\frac{n}{2}-m+\frac{1}{2}+j}} |f(y)|\,dy \\
&\le C\int_{y\in\Omega:\,|x-y|\le N^{-1}d(y)} d(y)^{m-\frac{n}{2}-\frac{1}{2}-j}|f(y)|\,dy \\
&\qquad + C\int_{y\in\Omega:\,|x-y|\ge N^{-1}d(y)} d(y)^{m-\frac{n}{2}+\frac{1}{2}-j}\,\frac{|f(y)|}{|x-y|}\,dy \\
&\le C\int_\Omega d(y)^{m-\frac{n}{2}+\frac{1}{2}-j}\,\frac{|f(y)|}{|x-y|}\,dy,
\end{aligned} \tag{4.10}$$

as desired.

This finishes the proof of (4.2) and (4.3) follows from it via the mapping properties of the Riesz potential. □

Proposition 4.1 has a natural analogue for the case when the dimension is even. The details are as follows.

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $n \in [2, 2m] \cap \mathbb{N}$ is even. Consider the boundary value problem*

$$(-\Delta)^m u = \sum_{|\alpha|\le m-\frac{n}{2}} c_\alpha \partial^\alpha f_\alpha \in W^{-m,2}(\Omega), \quad u \in \mathring{W}^{m,2}(\Omega). \tag{4.11}$$

Then the solution satisfies the estimate

$$|\nabla^{m-\frac{n}{2}} u(x)| \le C \sum_{|\alpha|\le m-\frac{n}{2}} \int_\Omega d(y)^{m-\frac{n}{2}-|\alpha|} \log\left(1+\frac{d(y)}{|x-y|}\right) |f_\alpha(y)|\,dy, \tag{4.12}$$

for all $x \in \Omega$, whenever the integrals on the right-hand side of (4.12) are finite. The constant C in (4.12) depends on m and n only.

In particular, for every $\varepsilon \in (0,1)$ there exists a constant $C_{\Omega,\varepsilon} > 0$ depending on m, n, ε and the domain Ω such that

$$\|\nabla^{m-\frac{n}{2}} u\|_{L^\infty(\Omega)} \leq C_{\Omega,\varepsilon} \sum_{|\alpha|\leq m-\frac{n}{2}} \left\| d(y)^{m-\frac{n}{2}-|\alpha|+\varepsilon} f_\alpha \right\|_{L^p(\Omega)}, \tag{4.13}$$

for all $p > \frac{n}{n-\varepsilon}$, provided that the norms on the right-hand side of (4.13) *are finite.*

Proof. The argument is fairly close to the proof of Proposition 4.1. We write

$$\left|\nabla^{m-\frac{n}{2}} u(x)\right| \leq C \sum_{|\alpha|\leq m-\frac{n}{2}} \int_\Omega \left|\nabla_x^{m-\frac{n}{2}} \nabla_y^{|\alpha|} G(x,y)\right| |f_\alpha(y)|\, dy, \tag{4.14}$$

for every $x \in \Omega$, and split the integrals on the right-hand side according to whether $|x-y| \leq N^{-1}d(y)$ or $|x-y| \geq N^{-1}d(y)$, $N \geq 25$. Then using (4.8) and (4.9) we bound each term on the right-hand side of (4.14) by

$$\begin{aligned} &C \int_{y\in\Omega:\,|x-y|\leq N^{-1}d(y)} \frac{1}{|x-y|^{\frac{n}{2}-m+|\alpha|}} \left(\frac{|x-y|}{d(y)}\right)^{\frac{n}{2}-m+|\alpha|} \\ &\qquad \times \log\left(1+\frac{d(y)}{|x-y|}\right) |f_\alpha(y)|\, dy \\ &+ C \int_{y\in\Omega:\,|x-y|\geq N^{-1}d(y)} \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}-|\alpha|} \frac{1}{|x-y|^{\frac{n}{2}-m+|\alpha|}} \\ &\qquad \times \log\left(1+\frac{\min\{d(y),d(x)\}}{|x-y|}\right) |f_\alpha(y)|\, dy. \end{aligned} \tag{4.15}$$

However, if $|x-y| \geq N^{-1}d(y)$ and hence, $d(x) \leq (N+1)|x-y|$, we have

$$\log\left(1+\frac{\min\{d(y),d(x)\}}{|x-y|}\right) \approx C \approx \log\left(1+\frac{d(y)}{|x-y|}\right). \tag{4.16}$$

Therefore, both terms in (4.15) are bounded by

$$C \int_\Omega d(y)^{m-\frac{n}{2}-j} \log\left(1+\frac{d(y)}{|x-y|}\right) |f_\alpha(y)|\, dy, \tag{4.17}$$

which leads to (4.12).

Finally, for every $0 < \varepsilon < 1$ there is $C_\varepsilon > 0$ such that $\log(1+x) \leq C_\varepsilon x^\varepsilon$, $x > 0$, which implies that

$$|\nabla^{m-\frac{n}{2}} u(x)| \leq C_\varepsilon \sum_{|\alpha|\leq m-\frac{n}{2}} \int_\Omega d(y)^{m-\frac{n}{2}-|\alpha|} \left(\frac{d(y)}{|x-y|}\right)^\varepsilon |f_\alpha(y)|\, dy, \tag{4.18}$$

for all $x \in \Omega$, $0 < \varepsilon < 1$.

Then, by the mapping properties of the Riesz potential we recover an estimate

$$\|\nabla^{m-\frac{n}{2}} u\|_{L^\infty(\Omega)} \leq C_{\Omega,\varepsilon} \sum_{|\alpha|\leq m-\frac{n}{2}} \left\| d(y)^{m-\frac{n}{2}-|\alpha|+\varepsilon} f_\alpha \right\|_{L^p(\Omega)},\ p > \frac{n}{n-\varepsilon}, \tag{4.19}$$

which leads to (4.13). □

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On Elliptic Operators in Nondivergence and in Double Divergence Form

Robert McOwen

To Vladimir Maz'ya for his 70th birthday

Abstract. This work surveys results on the existence and asymptotic behavior of the fundamental solution for an elliptic operator $\mathcal{L}$ in nondivergence form, including recent results for operators whose coefficients are continuous with mild conditions on the modulus of continuity: if the square of the modulus of continuity satisfies the Dini condition, then there is an integral invariant which controls the behavior of solutions of $\mathcal{L}^* u = 0$ and whether there is a fundamental solution for $\mathcal{L}$ that is asymptotic to the fundamental solution for the associated constant coefficient operator.

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1. Introduction

Consider an elliptic operator in nondivergence form

$$\mathcal{L}(x, \partial_x)\, u(x) := a_{ij}(x)\, \partial_i \partial_j u(x), \tag{1.1}$$

where $\partial_i = \partial/\partial x_i$ and we have used the summation convention for repeated indices. The coefficients $a_{ij} = a_{ji}$ are bounded, measurable functions defined on $\mathbb{R}^n$ for $n \geq 3$. (The case $n = 2$ can be treated similarly but some technical difficulties occur in the statement and proofs of results.) For convenience, we assume that the a_{ij} are real valued, although many results described in this paper (in particular, those of [12] and [13]) may be generalized to complex-valued coefficients. Under the assumption of real-valued coefficients, let us introduce the formal adjoint

$$\mathcal{L}(x, \partial_x)^*\, u(x) := \partial_i \partial_j (\, a_{ij}(x) u(x)), \tag{1.2}$$

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which might be described as being in "double divergence" form. We are interested in the fundamental solution $F(x,y)$ for $\mathcal{L}$, and its asymptotic behavior near $x=y$, under weak regularity assumptions on the coefficients a_{ij}. For an open set U, the fundamental solution is generally defined by the familiar equation

$$-\mathcal{L}(x,\partial_x)\,F(x,y)=\delta(x-y)\qquad \text{for } x,y\in U. \tag{1.3}$$

However, for nonsmooth coefficients, this may not be meaningful: $\partial_i\partial_j F(\cdot,y)$ is not in $L^1_{\mathrm{loc}}(U)$ and the coefficients a_{ij} are not sufficiently regular to allow the derivatives to be interpreted distributionally.

Recall that these difficulties do not occur when the elliptic operator is in divergence form: if we let

$$\mathcal{L}_0(x,\partial_x)u:=\partial_i(a_{ij}(x)\partial_j u(x)), \tag{1.4}$$

then $-\mathcal{L}_0(x,\partial_x)F_0(x,y)=\delta(x-y)$ means

$$\phi(y)=\int a_{ij}(x,y)\partial_j F_0(x,y)\partial_i\phi(x)\,dx\qquad \text{for } \phi\in C_0^\infty(U), \tag{1.5}$$

which only requires $\partial_j F_0\in L^1_{\mathrm{loc}}(U)$. In fact, if $U=B_1$ is the unit ball centered at the origin and $\mathcal{L}_0$ is uniformly elliptic in B_1, then it is well known (cf. [10]) that there is a unique Green's function $G_0(x,y)$, i.e., a solution of

$$-\mathcal{L}_0(x,\partial_x)G_0(x,y)=\delta(x-y)\text{ in } B_1,\quad G_0(x,y)=0 \text{ for } x\in\partial B_1, \tag{1.6}$$

which has the properties that it is nonnegative, satisfies $g_0(x,y)=g_0(y,x)$, and admits the upper and lower bounds

$$c_1|x-y|^{2-n}\le G_0(x,y)\le c_2|x-y|^{2-n}\qquad \text{for } x,y\in B_{1/2}. \tag{1.7}$$

Moreover, the Green's operator

$$\mathcal{G}_0\phi(x):=-\int G_0(x,y)\phi(y)\,dy\qquad \text{for } \phi\in C_0^\infty(U) \tag{1.8}$$

defines a left and right inverse for $\mathcal{L}_0$, i.e., $\mathcal{G}_0\mathcal{L}_0\phi=\phi=\mathcal{L}_0\mathcal{G}_0\phi$ for $\phi\in C_0^\infty(U)$.

For $\mathcal{L}$ in nondivergence form, even if the coefficients are not sufficiently regular for (1.3) to be well defined, the adjoint condition $-\mathcal{L}(y,\partial_y)^*\,F(x,y)=\delta(x-y)$ is well defined by

$$\phi(x)=-\int F(x,y)\mathcal{L}(y,\partial_y)\phi(y)\,dy\qquad \text{for } \phi\in C_0^\infty(U), \tag{1.9}$$

which only requires $F(x,\cdot)\in L^1_{\mathrm{loc}}(U)$ for fixed x. However, there are several complications that did not occur in the divergence form case. For example, let us consider the Green's function $G(x,y)$ on $U=B_1$, i.e.,

$$\phi(x)=-\int G(x,y)\mathcal{L}(y,\partial_y)\phi(y)\,dy\quad \text{for } \phi\in C^2(\overline{B_1}) \text{ with } \phi=0 \text{ on } \partial B_1. \tag{1.10}$$

If the coefficients are discontinuous, then $g(x,y)$ is not necessarily uniquely defined (cf. [4]). If the coefficients are continuous on $\overline{B}_1$, then $G(x,y)$ is uniquely defined and the Green's operator defines a left-inverse for $\mathcal{L}$, but $G(x,y)$ need not be locally bounded away from the diagonal $x = y$ (cf. [2]). Nevertheless, there are estimates as $y \to x$ that essentially generalize (1.7). To give a more specific statement of these estimates, let us fix $x = 0$ and let $g_1(y) = G(0, y)$. Then Bauman ([3]) found pointwise estimates for $g_1(y)$ in terms of a solution of $\mathcal{L}^* W = 0$: for $|y| \le 1/2$

$$c_1 \int_{|y|}^{1} \frac{s\,ds}{W(B_s)}\, W(y) \;\le g_1(y) \le c_2 \int_{|y|}^{1} \frac{s\,ds}{W(B_s)}\, W(y), \tag{1.11}$$

where $W(y) = g(x_0, y)$ with x_0 fixed in B_1 and $W(B_s) = \int_{B_s} W(y)\,dy$. If the coefficients a_{ij} are sufficiently regular, then W is positive and bounded, so $c_3 s^n \le W(B_s) \le c_4 s^n$ and the estimate (1.11) is comparable to (1.7). In general, W need not be bounded above or away from zero, but W cannot blow up or vanish too rapidly; Escauriaza ([5]) shows that for every $\gamma > 0$ there exists $N_\gamma > 0$ such that

$$\frac{r^{n+\gamma}}{N_\gamma} \le W(B_r) \le N_\gamma r^{n-\gamma}. \tag{1.12}$$

Now let us consider conditions under which a fundamental solution for $\mathcal{L}$ exists in the sense of (1.3). Let us review what is known in the classical case that the coefficients are λ-Hölder continuous in a bounded domain U where $\lambda \in (0,1)$. In this case, it is well known (cf.[16]) that $F(x, y)$ exists and is asymptotic as $x \to y$ to the fundamental solution for the constant coefficient operator $\mathcal{L}(y, \partial_x) = a_{ij}(y)\partial_i\partial_j$, obtained by freezing the coefficients a_{ij} at y. More specifically, we can write

$$F(x, y) = \widetilde{F}_y(x - y)(1 + H(x, y)), \tag{1.13}$$

where, letting $\mathbf{A}_y$ denote the matrix $a_{ij}(y)$ and $\langle , \rangle$ denote inner product in $\mathbb{R}^n$,

$$\widetilde{F}_y(x) := \frac{\langle \mathbf{A}_y^{-1} x, x\rangle^{\frac{2-n}{2}}}{(n-2)\,|S^{n-1}|\,\sqrt{\det \mathbf{A}_y}} \tag{1.14}$$

is the fundamental solution for $\mathcal{L}(y, \partial_x)$ and the remainder term $H(x, y)$ in (1.13) satisfies

$$|H(x, y)| + r|D_x H(x, y)| + r^2 |D_x^2 H(x, y)| \le c\, r^\lambda \quad \text{as} \quad r = |x - y| \to 0, \tag{1.15}$$

for all y in a compact subset of U. Using this fundamental solution, the operator

$$\mathcal{Q}\phi(x) := -\int_U F(x, y)\phi(y)\,dy \tag{1.16}$$

defines a right inverse for $\mathcal{L}$. (In fact, the existence of a local right inverse $\mathcal{Q}$ is true if the coefficients are only continuous, cf. [9], but continuity does not provide sufficient control for an asymptotic description such as (1.13)–(1.15).)

To generalize the classical case, let us allow the coefficients to have a weaker modulus of continuity, i.e., $a_{ij} \in C^\omega(U)$ where $\omega(r)$ is a continuous, nondecreasing function for $0 \le r < 1$ satisfying $\omega(0) = 0$ but vanishing more slowly than r as $r \to 0$ (i.e., $\omega(r)\, r^{-1+\kappa}$ is nonincreasing for some $\kappa > 0$), and

$$C^\omega(U) = \{f \in C(U) : |f(x) - f(y)| \le c\,\omega(|x - y|) \quad \text{for } x, y \in U\}. \tag{1.17}$$

If ω satisfies the Dini condition at zero, i.e., $\int_0^1 \omega(t)t^{-1}\,dt < \infty$, then the coefficients are "Dini continuous," and there are local regularity results analogous to the case of Hölder continuity (cf. Proposition 1.14 in Chapter 3 of [17]); there are also results on the asymptotics of fundamental solutions (cf. Section 3 in [8]) but only for operators in divergence form. However, we shall assume that the coefficients have modulus of continuity ω satisfying the "square-Dini condition"

$$\int_0^1 \omega^2(t)\,\frac{dt}{t} < \infty. \tag{1.18}$$

Obviously, (1.18) generalizes the Dini condition; for example, $\omega(r) = (1-\log r)^{-\beta}$ satisfies the Dini condition for $\beta > 1$ but the square-Dini condition for $\beta > 1/2$.

In the recent papers [12] and [13], it is found that the condition (1.18) is sufficient to obtain existence results and asymptotic descriptions for key solutions of both $\mathcal{L}(x,\partial_x)^* Z_y(x) = 0$ for $|x-y|<\epsilon$ and $\mathcal{L}(x,\partial_x)F(x,y) = 0$ for $0<|x-y|<\epsilon$. In both cases, the asymptotic behavior as $x \to y$ is controlled by the same integral term $I(x,y)$. This integral term $I(x,y)$ is most conveniently expressed when we use coordinates x in which y corresponds to $x = 0$ and $a_{ij}(0) = \delta_{ij}$; of course, this latter condition can always be achieved when the coefficients are real valued. In this case, we find that $I(x,0) = I(|x|)$ where

$$I(r) = \frac{1}{|S^{n-1}|}\int_{r<|z|<\epsilon}\left(\mathrm{tr}(\mathbf{A}_z) - n\,\frac{\langle \mathbf{A}_z z, z\rangle}{|z|^2}\right)\frac{dz}{|z|^n}. \tag{1.19}$$

If the coefficients are Dini continuous, then it is easy to see that $\lim_{r\to 0^+} I(r)$ exists and is finite, but this conclusion could be true even if the coefficients are not Dini continuous. In general we could have $I(r)$ unbounded as $r \to 0$, but it is not difficult to see that for every $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$|I(r)| \le \lambda|\log r| + C_\lambda \quad \text{for } 0 < r < \epsilon. \tag{1.20}$$

The results of [12] and [13] are both formulated using L^p-means. Let us fix $y \in \mathbb{R}^n$ and $p \in (1,\infty)$ and define

$$M_p(w,r;y) := \left(\fint_{A_r(y)} |w(x)|^p\,dx\right)^{1/p}, \tag{1.21}$$

where $A_r(y) = \{x : r < |x-y| < 2r\}$; here (and elsewhere in this paper) the slashed integral denotes mean value. If $y = 0$ then we denote $M_p(w,r;0)$ simply by $M_p(w,r)$. The main results of [12] can be expressed in the following:

Theorem 1.1. *Suppose* $|a_{ij}(x) - \delta_{ij}| \le \omega(|x|)$ *as* $|x| \to 0$ *where* ω *satisfies* (1.18). *For* $p \in (1,\infty)$ *and* $\epsilon > 0$ *sufficiently small, there exists a weak solution* $Z \in L^p_{\mathrm{loc}}(B_\epsilon)$ *of*

$$\mathcal{L}(x,\partial_x)^* Z = 0 \quad \text{in } B_\epsilon \tag{1.22}$$

satisfying

$$Z(x) = e^{-I(|x|)}(1+\zeta(x)), \tag{1.23}$$

where the remainder term ζ *satisfies*

$$M_p(\zeta,r) \le c\,\max\left(\omega(r),\sigma(r)\right) \quad \textit{with } \sigma(r) := \int_0^r \frac{\omega^2(t)}{t}\,dt. \tag{1.24}$$

Moreover, if $u \in L^p_{\text{loc}}(\overline{B}_\epsilon\backslash\{0\})$ *is a weak solution of* $\mathcal{L}(x,\partial_x)^* u = 0$ *in* B_ϵ *subject to the growth condition* $M_p(u,r) \le c\,r^{2-n+\epsilon_0}$ *where* $\epsilon_0 > 0$*, then there exists a constant* C *(depending on* u*) such that*

$$u(x) = C\,Z(x) + w(x) \tag{1.25}$$

where the remainder term w *satisfies* $M_p(w,r) \le c\,r^{1-\epsilon_1}$ *for* $0 < r < \epsilon$ *and any* $\epsilon_1 > 0$.

If the a_{ij} are real valued, then $a_{ij}(0) = \delta_{ij}$ can always be achieved by an affine change of variables, but Theorem 1.1 as stated also holds for complex-valued coefficients. Moreover, for real-valued a_{ij}, the solution Z may be scaled to coincide with the function W that occurs in (1.11), so the asymptotic behavior (1.23) implies

$$W(B_r) = c\,e^{-I(r)} r^n (1 + o(1)) \quad \text{as } r \to 0, \tag{1.26}$$

which is a refinement of (1.12), as can be seen from (1.20).

Now let us turn to the behavior of the fundamental solution as $x \to y$, and in particular whether (1.3) can be satisfied. For fixed y, this involves finding a solution of

$$\mathcal{L}\mathcal{Z}_y(x) = 0 \quad \text{for } 0 < |x-y| < \epsilon \tag{1.27}$$

with the appropriate singularity at $x = y$. If we again assume that $y = 0$ and $a_{ij}(0) = \delta_{ij}$, then the analysis in [13] shows that there exists a solution

$$\mathcal{Z}_0(x) \sim \frac{|x|^{2-n}}{n-2}\, e^{I(|x|)} \quad \text{as } |x| \to 0, \tag{1.28}$$

where $I(r)$ is given by (1.19). The behavior of $I(r)$ as $r \to 0$ not only controls the leading asymptotic of $\mathcal{Z}_0(x)$, but whether we can solve (1.3) at $y = 0$. There are three important cases to consider:

1. $I(0) = \lim_{r\to 0} I(r)$ *exists and is finite.*

In this case, $\mathcal{Z}_0(x)$ may be scaled by a constant multiple to make it asymptotic to the fundamental solution for the Laplacian. In fact, the distributional equation

$$-\mathcal{L}(x,\partial_x)\mathcal{Z}_0(x) = C_0\,\delta(x), \tag{1.29}$$

can be solved to find

$$C_0 = |S^{n-1}|\,e^{I(0)}, \tag{1.30}$$

where (1.29) is interpreted to mean

$$-\int_{|x|<\epsilon} \left[(a_{ij}(x) - \delta_{ij})\,\partial_i\partial_j\mathcal{Z}_0(x)\,\phi(x) - \partial_i\mathcal{Z}_0(x)\,\partial_j\phi(x)\right]\,dx = C_0\,\phi(0) \tag{1.31}$$

for all $\phi \in C_0^\infty(B_\epsilon)$; the convergence of the integral in (1.31) is part of the conclusion. Using Theorem 1.1, note that this case also allows $\gamma = 0$ in (1.12).

2. *$I(r) \to -\infty$ as $r \to 0$.*
We see that $\mathcal{Z}_0(x) = o(|x|^{2-n})$ as $|x| \to 0$, and we can solve (1.29) to find $C_0 = 0$. Thus, in this case, we obtain the interesting corollary that

$$\mathcal{L}u = 0 \quad \text{in } B_\epsilon \tag{1.32}$$

admits a solution $u = \mathcal{Z}_0$ that is quite singular at $x = 0$: $|\mathcal{Z}_0(x)| \geq C_\lambda |x|^{2-n+\lambda}$ for every $\lambda > 0$. In particular, local regularity of solutions of the homogeneous equation (1.32) does not hold.

3. *$I(r) \to \infty$ as $r \to 0$.*
Now we find $\mathcal{Z}_0(x)|x|^{n-2} \to \infty$ as $|x| \to 0$, so this solution grows more rapidly than the fundamental solution for the Laplacian. Although $\mathcal{Z}_0$ still satisfies (1.29), the integrals in (1.31) no longer converge and we can no longer solve for C_0.

In order to use $\mathcal{Z}_y$ to construct a fundamental solution for $\mathcal{L}$, we need to allow y to vary. For general coordinates x, we transform (1.19) to find (cf. [13])

$$I(x,y) = I_y(r) \quad \text{where} \quad r^2 = \langle \mathbf{A}_y^{-1}(x-y), (x-y) \rangle. \tag{1.33}$$

We now assume that $\lim_{x \to y} I(x,y) = \lim_{r \to 0} I_y(r) = I_y(0)$ exists and is finite at every $y \in U$ and that the convergence is uniform:

$$|I_y(r) - I_y(0)| \leq \theta(r) \quad \text{for all } y \in U, \tag{1.34}$$

where $\theta(r)$ is a continuous, nondecreasing function for $0 \leq r < 1$ satisfying $\theta(0) = 0$ but vanishing slower than r (i.e., $\theta(r)\, r^{-1+\nu}$ is nonincreasing for some $\nu > 0$). The main result of [13] asserts the existence of a fundamental solution $F(x,y)$ in the form (1.13) with remainder term $H(x,y) = H_y(x)$ estimated using ω, σ, and θ:

Theorem 1.2. *Suppose* (1.1) *is uniformly elliptic in a bounded open set U, where the coefficients a_{ij} are continuous functions with uniform modulus of continuity ω satisfying* (1.18)*. Suppose also that $I(x,y)$ satisfies* (1.34)*. Then there is a function $F(x,y)$ for $x, y \in U$ satisfying* (1.3) *and the asymptotic description* (1.13)*, where the remainder term $H(x,y)$ may be estimated as follows: for any $p \in (1,\infty)$ and any compact set $K \subset U$ we have*

$$M_p(H_y, r; y) + r^2 M_p(D^2 H_y, r; y) \leq c \max(\omega(r), \sigma(r), \theta(r)) \quad \text{as } r \to 0, \tag{1.35}$$

with constant c independent of $y \in K$.

Taking $p > n$, the Sobolev inequality implies that we have pointwise bounds on $|H(x,y)| + r|D_x H(x,y)|$ as $r = |x-y| \to 0$, but we no longer have pointwise bounds on $D_x^2 H(x,y)$ as we did in (1.15) when the coefficients a_{ij} were Hölder continuous.

In conclusion, we see that the integral invariant $I(x,y)$ plays an important role for the existence and asymptotics of the fundamental solution of an elliptic operator in nondivergence form when the coefficients are square Dini continuous but not Dini continuous. In particular, the behavior of $I(x,y)$ as $x \to y$ determines whether a local nonnegative solution of $\mathcal{L}^* Z = 0$ remains positive or bounded, and whether (1.3) admits a solution. This subtlety does not occur when the operator is in divergence form or the coefficients are Dini continuous.

2. Discussion of the proofs

One basic idea lies behind the proofs of both Theorems 1.1 and 1.2: use a "spectral decomposition" to split the solution u into a radial part $h(r)$ and a spherically symmetric part $v(x)$. This reduces the partial differential equation to an ordinary differential equation for h (that also depends on v) and a partial differential equation for v (that also depends on h). The ordinary differential equation for h can be solved (in terms of v) and used to eliminate h from the partial differential equation for v. Estimates for v can then be obtained by comparison with estimates for the Laplacian (since $a_{ij}(0) = \delta_{ij}$). This leads to an operator equation that can be solved to find v, and from this we obtain h. Now let us give a few more details behind each proof.

The spectral decomposition is defined using the spherical mean

$$\overline{u}(r) = \fint_{S^{n-1}} u(r\theta)\, ds, \tag{2.1}$$

where $r = |x| > 0$, $\theta = x/|x| \in S^{n-1}$, and ds denotes standard surface measure, to write

$$u(x) = h(|x|) + v(x) \quad \text{where } h(r) = \overline{u}(r) \text{ and } \overline{v}(r) = 0. \tag{2.2}$$

In the analysis of both theorems, we encounter

$$\alpha_n(r) := \fint_{S^{n-1}} a_{ii}(r\theta)\, ds \quad \text{and} \quad \alpha(r) := \fint_{S^{n-1}} a_{ij}(r\theta)\theta_i\theta_j\, ds \tag{2.3}$$

as well as

$$R(r) := n - \frac{\alpha_n(r)}{\alpha(r)}. \tag{2.4}$$

Notice that

$$|\alpha_n(r) - n| + |\alpha(r) - 1| + |R(r)| \le c\,\omega(r) \quad \text{for } 0 < r < \epsilon. \tag{2.5}$$

By rescaling, we can assume for convenience that $\epsilon = 1$ and that $\omega(r) \le \delta$ for $0 < r < 1$ where δ is as small as necessary. Moreover, let us extend the coefficients a_{ij} by δ_{ij} for $|x| > 1$, so that $\mathcal{L}$ is defined on all of $\mathbb{R}^n$ with $\alpha_n(r) \equiv n$, $\alpha(r) \equiv 1$, and $R(r) \equiv 0$ for $r > 1$. Moreover, the integral invariant I can be written

$$I(r) = -\int_r^1 R(\rho)\,\alpha(\rho)\,\frac{d\rho}{\rho}. \tag{2.6}$$

To produce the solution Z in Theorem 1.1, write $Z(x) = h(|x|) + v(x)$ as above and introduce

$$y(r) := \alpha(r)h(r) + \fint_{S^{n-1}} v(r\theta)(a_{ij}(r\theta)\theta_i\theta_j - 1)\, ds. \tag{2.7}$$

The equation $\mathcal{L}^* Z = 0$ on $\mathbb{R}^n$ leads to the ordinary differential equation

$$y'(r) + \frac{R(r)}{r} y(r) = \frac{1}{r} K v(r) \quad \text{for } r > 0, \tag{2.8}$$

where K maps functions of x to functions of r and is defined by

$$Kv(r) := \fint_{S^{n-1}} v(r\theta)a_{ii}(r\theta)\,ds - \frac{\alpha_n(r)}{\alpha(r)}\fint_{S^{n-1}} v(r\theta)a_{ij}(r\theta)\theta_i\theta_j\,ds, \tag{2.9}$$

which shows that

$$M_p(Kv,r) \le c\,\omega(r)\,M_p(v,r). \tag{2.10}$$

Additional control on v is obtained from the partial differential equation on $\mathbb{R}^n$

$$\Delta v = \overline{\partial_i\partial_j(B_{ij}(v))} - \partial_i\partial_j(B_{ij}(v)) + \overline{\partial_i\partial_j(\phi_{ij}y)} - \partial_i\partial_j(\phi_{ij}y), \tag{2.11}$$

where

$$B_{ij}(v)(x) := (a_{ij}(x)-\delta_{ij})\left(v(x) - \frac{1}{\alpha(r)}\fint_{\partial B_1} v(r\theta)(a_{ij}(r\theta)\theta_i\theta_j - 1)\,ds\right) \tag{2.12}$$

and

$$\phi_{ij}(x) := \frac{a_{ij}(x)-\delta_{ij}}{\alpha(|x|)}. \tag{2.13}$$

Using the integrating factor

$$E(r) = \exp\left(\int_r^\infty R(t)\,t^{-1}\,dt\right) = \exp\left(\int_r^1 R(t)\,t^{-1}\,dt\right) \tag{2.14}$$

to solve (2.8) for y in terms of v, the y in (2.11) can be eliminated. Inverting the Laplacian results in an operator equation for v, whose unique solution is found in [12] to satisfy

$$M_p(v,r) \le \begin{cases} c\,\omega(r)\,E(r) & \text{for } 0<r<1, \\ c\,r^{-n} & \text{for } r>1. \end{cases} \tag{2.15}$$

These estimates show that for $0<r<1$ we have

$$y(r) = c_1\,E(r) + \zeta_0(r) \text{ where } |\zeta_0(r)| \le c\max(\omega(r),\sigma(r)), \tag{2.16}$$

where c_1 is a constant. But then (2.5) and (2.7) show that a similar estimate holds with y replaced by h. Finally, (2.6) shows that

$$E(r) = C\,e^{-I(r)}(1+\zeta_1(r)), \quad \text{where } |\zeta_1(r)| \le c\,\sigma(r). \tag{2.17}$$

We can now rescale $Z = h+v$ to obtain the desired asymptotic form (1.23).

The "uniqueness" statement (1.25) is obtained using isomorphism theorems on weighted Sobolev spaces. Of course, we are only interested in the behavior of solutions in a neighborhood of $x=0$, but the fact that v was constructed on $\mathbb{R}^n$ enables us to avoid imposing boundary conditions by imposing independent weights at $x=0$ and $x=\infty$. Effectively, this puts the analysis into the theory of operators on cylinders, as studied in [14] and [11], but for the special case of small perturbations of the Laplacian, which is quite well understood (cf. [15]).

Now let us consider the proof of Theorem 1.2. This proceeds in three steps:
Step 1. For fixed y we construct a solution $\mathcal{Z}_y(x)$ of

$$\mathcal{L}(x,\partial_x)\mathcal{Z}_y(x) = 0 \quad \text{for } x \in B_\epsilon(y)\backslash\{y\}. \tag{2.18}$$

Step 2. For fixed y we investigate when we have

$$-\mathcal{L}(x,\partial_x)\mathcal{Z}_y(x) = C_y\delta(x-y) \quad \text{for } x \in B_\epsilon(y) \tag{2.19}$$

with a computable constant C_y.

Step 3. We construct our fundamental solution in such a way that for x sufficiently close to y we have

$$F(x,y) = \mathcal{Z}_y(x)/C_y + v(x,y), \tag{2.20}$$

where $\mathcal{L}(x,\partial_x)v(x,y) = 0$.

The construction in Step 1 is analogous to the construction of the solution in Theorem 1.1. As before, let us fix $y = 0$, assume $a_{ij}(0) = \delta_{ij}$, and use the spectral decomposition (2.2) to write our desired solution as $\mathcal{Z}(x) = h(|x|) + v(x)$. We find that h satisfies the ordinary differential equation

$$\alpha(r)h'' + \frac{\alpha_n(r)-\alpha(r)}{r}h' + \overline{\beta_{ij}\partial_i\partial_j v}(r) = 0, \tag{2.21}$$

and v satisfies the partial differential equation

$$-\Delta v = \beta_{ij}\partial_i\partial_j h - \overline{\beta_{ij}\partial_i\partial_j h} + \beta_{ij}\partial_i\partial_j v - \overline{\beta_{ij}\partial_i\partial_j v}, \tag{2.22}$$

where $\beta_{ij}(x) = a_{ij}(x) - \delta_{ij}$ satisfies $|\beta_{ij}(x)| \le \omega(|x|)$ for $|x| < 1$ and $\beta_{ij}(x) = 0$ for $|x| > 1$. The strategy is as before: solve (2.21) for h (in terms of v), substitute that into (2.22), and invert the Laplacian to obtain an operator equation for v. However, now we need to control the derivatives of v in L^p so we introduce

$$M_{2,p}(v,r) := r^2 M_p(D^2 v, r) + rM_p(Dv,r) + M_p(v,r). \tag{2.23}$$

It is shown in [13] that the unique solution v satisfies

$$M_{2,p}(v,r) \le \begin{cases} c\,r^{2-n}\,\omega(r)\,E^{-1}(r) & \text{for } 0 < r < 1, \\ c\,r^{1-n} & \text{for } r > 1. \end{cases} \tag{2.24}$$

In terms of this v, the solution of (2.21) can be written as

$$h(r) = \int_r^1 s^{1-n} e^{I(s)}\,ds\,(1+\zeta(r)) \tag{2.25}$$

where $M_{2,p}(\zeta,r) \le c\,\max(\omega(r),\sigma(r))$. Integration by parts in (2.25) together with (2.24) and the Sobolev inequality shows that the solution $\mathcal{Z} = h + v$ satisfies

$$\mathcal{Z}(x) = \frac{|x|^{2-n}e^{I(|x|)}}{n-2}\,(1+\xi(x)) \tag{2.26}$$

where

$$rM_\infty(D\xi,r) \le c\,\max(\omega(r),\sigma(r)) \tag{2.27}$$

with $M_\infty(w,r)$ defined using essmax in place of L^p in (1.21); however, we do not obtain estimates on the second-order derivatives of ξ in (2.26). This completes Step 1.

For Step 2, the above description of $\mathcal{Z}$ implies that for every $\mu > 0$ we have

$$M_p(\partial_i\partial_j\mathcal{Z},r) \le C_\mu\,r^{-n-\mu} \quad \text{for } 0 < r < 1. \tag{2.28}$$

Since the a_{ij} are bounded, this implies that $\mathcal{L}\mathcal{Z}$ can be regularized at $x = 0$ to give a distribution that extends to a continuous linear functional on Hölder continuous functions. But $\mathcal{L}\mathcal{Z} = 0$ for $0 < |x| < 1$ means that $\mathcal{L}\mathcal{Z} = C_0\,\delta$ for some constant C_0, and we can determine C_0 if we can compute

$$C_0 = \int_{|x|<1} [(-a_{ij}(x) + \delta_{ij})\partial_i\partial_j\mathcal{Z}(x)\,\phi(|x|) + \partial_i\mathcal{Z}(x)\partial_j\phi(|x|)]\,dx \tag{2.29}$$

for some $\phi \in C_0^\infty(B_1)$ with $\phi(0) = 1$. In [13] it is shown that if $I(0) = \lim_{r\to 0} I(r)$ exists and is finite then the integral in (2.29) converges and we can calculate the limit to obtain

$$C_0 = |S^{n-1}|e^{I(0)}, \tag{2.30}$$

which is the value that we expect if $\mathcal{Z}$ as in (2.26) is to be asymptotic to the fundamental solution for Δ. In fact, the assumption that $I(0)$ exists and is finite also allows us to improve the estimate on the error term ξ in the asymptotic description of $\mathcal{Z}(x)$ given in (2.26); if we assume that

$$|I(r) - I(0)| \le \theta(r) \tag{2.31}$$

where $\theta(r)$ is a continuous, nondecreasing function for $0 \le r < 1$ satisfying $\theta(0) = 0$ and $\theta(r)\,r^{-1+\nu}$ is nonincreasing for some $\nu > 0$, then we can improve (2.27) to estimate second-order derivatives of ξ in L^p:

$$r^2 M_p(D^2\xi, r) \le c\,\max(\omega(r), \sigma(r), \theta(r)). \tag{2.32}$$

Finally, for Step 3 we want to allow $y \ne 0$ so we need to assume $I_y(0)$ exists and is finite for every $y \in U$; in fact, the assumption (1.34) enables us to obtain estimates that are uniform over U. As in [13], it is convenient to construct F as the restriction to U of the Green's function $G(x, y)$ for a smooth bounded domain V which contains $\overline{U}$. In fact, the a_{ij} may be extended to $\overline{V}$ with the same modulus of continuity $\omega(r)$, and for every $y \in V$ we can construct $Z_y(x)$ and C_y in a small ball B_ϵ where ϵ may be taken independent of $y \in \overline{V}$. Now define

$$G(x, y) := \eta_y(|x - y|)\mathcal{Z}_y(x)/C_y + v(x, y) \tag{2.33}$$

where $\eta_y(|\cdot - y|) \in C_0^\infty(V)$ with $\eta_y(r) \equiv 1$ for r sufficiently small, and $v(x, y)$ is the solution of the Dirichlet problem

$$\begin{aligned} &\mathcal{L}(x, \partial_x)v(x, y) = \psi(x, y) \quad \text{for } x \in V, \\ &v(x, y) = 0 \quad \text{for } x \in \partial V, \end{aligned} \tag{2.34}$$

where $\psi(x, y) := -[\mathcal{L}(x, \partial_x), \eta_y(x)]\mathcal{Z}_y(x)/C_y$ is in $L^p(V)$ for fixed $y \in V$. Letting $F(x, y) = G(x, y)$ for $x, y \in U$ defines the desired fundamental solution, completing the proof.

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On the Well-posedness of the Dirichlet Problem in Certain Classes of Nontangentially Accessible Domains

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Dedicated with great pleasure to Vladimir Maz'ya on the occasion of his 70th birthday.

Abstract. We prove that if $\Omega \subset \mathbb{R}^n$ is a bounded NTA domain (in the sense of Jerison and Kenig) with an Ahlfors regular boundary, and which satisfies a uniform exterior ball condition, then the Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega, \qquad u\big|_{\partial\Omega} = f \in L^p(\partial\Omega, d\sigma),$$

has a unique solution for any $p \in (1, \infty)$. This solution satisfies natural non-tangential maximal function estimates and can be represented as

$$u(y) = -\int_{\partial\Omega} \partial_{\nu(x)} G(x, y) f(x)\, d\sigma(x), \qquad y \in \Omega.$$

Above, ν denotes the outward unit normal to Ω and $G(\cdot, \cdot)$ stands for the Green function associated with Ω.

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1. Introduction

This paper deals with the issue of the well-posedness of the Dirichlet problem for the Laplacian in bounded NTA subdomains of $\mathbb{R}^n$ satisfying additional properties (to be specified below). For this problem we seek solutions whose nontangential maximal functions are pth power integrable, and which attain their boundary data in the sense of a.e. nontangential convergence. In order to be more precise, we need some notation.

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Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded open set. As surface measure, σ, we take the restriction of the $(n-1)$-dimensional Hausdorff measure to $\partial\Omega$, the topological boundary of Ω. That is, $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$, as a Boreal measure, and we let $L^p(\partial\Omega, d\sigma)$, $0 < p \leq \infty$, denote the Lebesgue scale on $\partial\Omega$. Next, fix $\kappa > 0$ and for each boundary point $z \in \partial\Omega$ introduce the non-tangential approach region

$$\Gamma_\kappa(z) := \{x \in \Omega : |x - z| < (1+\kappa) \operatorname{dist}(x, \partial\Omega)\}. \tag{1.1}$$

Generally speaking, given a function $u : \Omega \to \mathbb{R}$, define the nontangential trace of u on $\partial\Omega$ by

$$u\Big|_{\partial\Omega}(z) := \lim_{\substack{x \to z \\ x \in \Gamma_\alpha(z)}} u(x), \qquad z \in \partial\Omega, \tag{1.2}$$

and the nontangential maximal function of u by

$$(\mathcal{N}_\kappa u)(z) := \sup \{|u(x)| : x \in \Gamma_\kappa(z)\}, \qquad z \in \partial\Omega. \tag{1.3}$$

Then the Dirichlet problem alluded to above reads

$$(D)_p \begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u\Big|_{\partial\Omega} = f \text{ on } \partial\Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, d\sigma). \end{cases} \tag{1.4}$$

For this problem, a solution is sought for which

$$\|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega, d\sigma)} \leq C(\Omega, \kappa, p) \|f\|_{L^p(\partial\Omega, d\sigma)}. \tag{1.5}$$

We seek to establish the unique solvability of $(D)_p$ assuming that Ω is a bounded NTA domain (cf. Definition 2.6), which satisfies certain additional conditions. The concept of NTA (acronym for nontangentially accessible) domain has been introduced by D. Jerison and C. Kenig in [14] in order to answer a question posed by E. Stein who has asked for the identification of the most general class of domains for which non-tangential behavior (of harmonic functions) is meaningful.

In the case when, in addition to being a bounded NTA domain, Ω has an Ahlfors regular boundary (cf. (2.5)), it turns out that $(D)_p$ is uniquely solvable, and the solution satisfies (1.5), provided p is large enough. More specifically, the following holds.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain, with an Ahlfors regular boundary. There exists $1 \leq p_* < \infty$ such that for all indices $p \in (p_*, \infty)$, the Dirichlet problem* (1.4) *has a unique solution which, in addition, satisfies* (1.5).

This is an extension of Theorem 10.1 from [14], which deals with the special case when Ω is a bounded BMO_1 domain in $\mathbb{R}^n$. This result, essentially due to D. Jerison, C. Kenig and G. David, is proved in § 4 by combining the approach in [14] with a result proved in [6], to the effect that the harmonic measure ω^x of Ω with pole at a fixed point $x \in \Omega$ belongs to $A_\infty(d\sigma)$ (i.e., a scale invariant version of mutual absolute continuity). See § 4 for a more detailed discussion.

The main result of this paper regards the well-posedness of the Dirichlet problem in the class of bounded NTA domains with Ahlfors regular boundaries, and which satisfy a uniform exterior ball condition (UEBC, for short). The latter property means that there exists a number $r > 0$ such that at each boundary point x it is possible to find a an open ball B_r which is contained in the complement of the domain and such that $x \in \partial B_r$. (The concept of uniform interior ball condition – UIBC, for short – is defined analogously, working with the complement of the domain.) Specifically, we have the following theorem.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain with an Ahlfors regular boundary, and which satisfies a UEBC. Denote by ν the outward unit normal to Ω and let $G(\cdot,\cdot)$ stand for the Green function associated with Ω. Then for any $p \in (1,\infty)$ the Dirichlet problem* (1.4) *has a unique solution. This solution satisfies* (1.5)*, with a constant C which depends only on the NTA, Ahlfors and UEBC character of Ω, and can be expressed as*

$$u(y) = -\int_{\partial\Omega} \partial_{\nu(x)} G(x,y) f(x)\, d\sigma(x), \qquad y \in \Omega. \tag{1.6}$$

Thus, in the setting of Theorem 1.1, the critical exponent $p_* \in [1,\infty)$ can be taken to be 1 if Ω also satisfies a UEBC. In order to further place this result into perspective, let us record the following known results:

$$\Omega \text{ bounded } C^1 \text{ domain} \Longrightarrow p_* = 1; \ [10] \tag{1.7}$$

$$\Omega \text{ Lipschitz domain} \Longrightarrow p_* = 2 - \varepsilon, \text{ for some } \varepsilon = \varepsilon(\Omega) > 0; \ [5] \tag{1.8}$$

$$\Omega \text{ bounded regular SKT domain} \Longrightarrow p_* = 1; \ [13]. \tag{1.9}$$

In (1.9), a regular SKT (Semmes-Kenig-Toro) domain is a two-sided NTA domain, Ω, with an Ahlfors regular boundary, and such that its unit normal ν belongs to $\mathrm{VMO}\,(\partial\Omega, d\sigma)$, Sarason's space of functions of vanishing mean oscillations.

The uniform (interior/exterior) ball condition has been originally introduced by Poincaré in [18] and, through the many intervening years, it has proved to be a natural, useful condition in the study of the boundary behavior of solutions of large classes of elliptic differential equations. While it can be shown that

$$\Omega \subset \mathbb{R}^n \text{ satisfies a UIBC and a UEBC} \Longleftrightarrow \partial\Omega \in C^{1,1}, \tag{1.10}$$

a mere one-sided ball condition permits the boundary of the domain in question to be quite rough. For instance, any bounded convex domain is NTA, with Ahlfors regular boundary, and satisfies a UEBC (for every $r > 0$). Theorem 1.2 also applies to the class of bounded Lipschitz domains which satisfies a UEBC.

In contrast to [10], [5], [13] where (1.7)–(1.9) have been proved by means of boundary layer potentials, in this paper we establish the well-posedness result stated in Theorem 1.2 by relying on the Green representation formula (1.6). A key ingredient in this regard is the Green function estimate

$$|\nabla_x G(x,y)| \le C \ \mathrm{dist}\,(y,\partial\Omega)|x-y|^{-n}, \qquad \forall\, x,y \in \Omega, \tag{1.11}$$

proved by M. Grüter and K.-O. Widman in [12], for domains $\Omega \subset \mathbb{R}^n$ satisfying a UEBC. The fact that $\partial\Omega$ is Ahlfors regular then allows us to use the Poisson kernel-like behavior of the right-hand side of (1.11) in order to conclude that the function u in (1.6) satisfies (1.5). This takes care of the existence part in Theorem 1.2. For the uniqueness part, we combine the approach in [10], [15], [13] with the estimates for the Green function from [14]. This portion of our analysis does not use the UEBC and, hence, applies to any bounded NTA domain with an Ahlfors regular boundary.

The organization of the remainder of the paper is as follows. In Section 2 we collect a number of useful definitions and background results pertaining to Ahlfors regularity, uniformly rectifiable domains, nontangential maximal operators, NTA domains, and a version of Gauss-Green formula recently proved in [13] which is well suited for the kind of functions we are dealing with here. Next, in Section 3, we review the construction and properties of harmonic measures in NTA domains, largely following the exposition in [14]. Here we also discuss the absolute continuity result of David-Jerison [6] and present a useful Fatou-type result. Section 4 contains a collection of properties of the Green function associated with domains of increased regularity (arbitrary domains, domains of class S, and domains satisfying a UEBC). In Section 5 we then present the proofs of Theorems 1.1–1.2. For the sake of clarity, we choose to treat separately the issues of existence and estimates (Theorem 5.1) and uniqueness (Theorem 5.3), given that the latter both makes use of, and holds in greater generality than, the former. Finally, in Section 6 we collect a number of known, auxiliary results which are used throughout the paper.

2. Analysis in uniformly rectifiable domains

Let $\Omega \subset \mathbb{R}^n$ be a fixed domain of locally finite perimeter. With $\mathbf{1}_E$ denoting the characteristic function of a set E, recall that this means that

$$\nabla \mathbf{1}_\Omega \text{ is a locally finite } \mathbb{R}^n\text{-valued measure.} \tag{2.1}$$

Essentially, domains of locally finite perimeter make up the largest class of domains for which some version of the classical Gauss-Green formula holds for smooth vector fields with compact support in $\mathbb{R}^n$. There are several excellent accounts on this topic, including the monographs by H. Federer [11], W. Ziemer [24], and L. Evans and R. Gariepy [9]. Here we wish to mention that if $\Omega \subset \mathbb{R}^n$ is a domain of locally finite perimeter, then there exists a suitably-defined concept of outward unit normal, which we denote by ν, and the role of the surface measure on $\partial\Omega$ is played by $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. Here and elsewhere, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. Recall that the measure-theoretic boundary $\partial_*\Omega$ of a domain $\Omega \subseteq \mathbb{R}^n$ is defined by

$$\begin{aligned} \partial_*\Omega := \Big\{ x \in \partial\Omega : \limsup_{r\to 0} \frac{\mathcal{H}^n(B(x,r)\cap\Omega)}{r^n} > 0, \\ \limsup_{r\to 0} \frac{\mathcal{H}^n(B(x,r)\setminus\Omega)}{r^n} > 0 \Big\}. \end{aligned} \tag{2.2}$$

As is well known, if Ω has locally finite perimeter, then the outward unit normal is defined σ-a.e. on $\partial_*\Omega$. In particular, if

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0, \tag{2.3}$$

then ν is defined σ-a.e. on $\partial\Omega$. Granted that Ω is a domain of locally finite perimeter, the Gauss-Green formula alluded to above reads

$$\int_{\Omega} \operatorname{div} v \, dx = \int_{\partial\Omega} \langle \nu, v \rangle \, d\mathcal{H}^{n-1}, \qquad \forall\, v \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n). \tag{2.4}$$

The conditions on the vector field v can be relaxed. For example, it is known that it suffices to assume that its components are compactly supported Lipschitz functions. However, for the purposes we have in mind, we require an even more refined version, discussed below in Theorem 2.12. To be able to state it requires a number of preliminaries.

Definition 2.1. A closed set $\Sigma \subset \mathbb{R}^n$ is called *Ahlfors regular* provided there exist $0 < C_1 \leq C_2 < \infty$ such that

$$C_1\, r^{n-1} \leq \mathcal{H}^{n-1}\big(B(x,r) \cap \Sigma\big) \leq C_2\, r^{n-1}, \tag{2.5}$$

for each $x \in \Sigma$ and $r \in (0,\infty)$ (if Σ is compact, (2.5) is required only for $r \in (0,1]$). The constants C_1, C_2 intervening in (2.5) will be referred to as the Ahlfors constants of $\partial\Omega$.

Finally, an open set $\Omega \subset \mathbb{R}^n$ is said to be an *Ahlfors regular domain* provided $\partial\Omega$ is Ahlfors regular.

It should be pointed out that Ahlfors regularity is not a regularity property per se, but rather a scale-invariant way of expressing the fact that the set in question is $(n-1)$-dimensional. Most of our analysis will be done on Ahlfors regular domains. Note that if (2.5) holds then (cf. Theorem 4 on p. 61 in [9]),

$$\sigma := \mathcal{H}^{n-1} \lfloor \Sigma \text{ is a Radon, doubling measure.} \tag{2.6}$$

Hence fundamental results of [3] (cf. the discussion on p. 624) yield the following.

Proposition 2.2. *An Ahlfors regular surface $\Sigma \subset \mathbb{R}^n$ is a space of homogeneous type (in the sense of Coifman-Weiss), when equipped with the Euclidean distance and the measure $\sigma = \mathcal{H}^{n-1}\lfloor\Sigma$. In particular, the associated Hardy-Littlewood maximal operator*

$$\mathcal{M}_\sigma f(x) := \sup_{r>0} \fint_{y \in \Sigma:\, |x-y|<r} |f(y)|\, d\sigma(y), \qquad x \in \Sigma, \tag{2.7}$$

is bounded on $L^p(\Sigma, d\sigma)$ for each $p \in (1,\infty)$. Here and elsewhere, the barred integral denotes averaging (with the convention that this is zero if the set in question has zero measure). Furthermore, there exists $C = C(\Sigma) \in (0,\infty)$ such that

$$\sigma\Big(\{x \in \Sigma :\, \mathcal{M}_\sigma f(x) > \lambda\}\Big) \leq C\, \lambda^{-1} \|f\|_{L^1(\Sigma, d\sigma)}, \tag{2.8}$$

for every $f \in L^1(\Sigma, d\sigma)$ and $\lambda > 0$.

We further elaborate on the properties of the Hardy-Littlewood maximal operator in § 5. For now, we briefly review the concept of uniform rectifiability. Following G. David and S. Semmes [7] we make the following definition.

Definition 2.3. Call $\Sigma \subset \mathbb{R}^n$ *uniformly rectifiable* provided it is Ahlfors regular and the following holds. There exist ε, $M \in (0,\infty)$ (called the UR constants of Σ) such that for each $x \in \Sigma$, $r > 0$, there is a Lipschitz map $\varphi : B_r^{n-1} \to \mathbb{R}^n$ (where B_r^{n-1} is a ball of radius r in $\mathbb{R}^{n-1}$) with Lipschitz constant $\leq M$, such that

$$\mathcal{H}^{n-1}\big(\Sigma \cap B(x,r) \cap \varphi(B_r^{n-1})\big) \geq \varepsilon r^{n-1}. \tag{2.9}$$

If Σ is compact, this is required only for $r \in (0,1]$.

An important class of uniformly rectifiable sets was identified in [6], where G. David and D. Jerison proved the following result.

Proposition 2.4. *Let $\Sigma \subset \mathbb{R}^n$ be closed and Ahlfors regular. Assume Σ satisfies the following "two disks" condition. There exists $C_0 \in (0,\infty)$ such that for each $x \in \Sigma$ and $r > 0$, there exist two $(n-1)$-dimensional disks, with centers at a distance $\leq r$ from x, radius r/C_0, contained in two different connected components of $\mathbb{R}^n \setminus \Sigma$. (If Σ is compact, one can pick $R_0 \in (0,\infty)$ and restrict attention to $r \in (0,R_0]$.) Then Σ is uniformly rectifiable.*

Following [13], we also make the following:

Definition 2.5. A nonempty, proper open subset Ω of $\mathbb{R}^n$ is called a *UR domain* provided $\partial\Omega$ is uniformly rectifiable and (2.3) holds.

We impose the last condition to eliminate such cases as a slit disk. Let us emphasize that, by definition, a UR domain Ω has an Ahlfors regular boundary.

One important class of UR domains is the class of Ahlfors regular domains with the NTA property, introduced in [14]. We recall the definition here, since these domains will play a role in subsequent sections.

Definition 2.6. A (bounded) open set $\Omega \subset \mathbb{R}^n$ is called an *NTA domain* (in the sense of Jerison and Kenig) provided Ω satisfies a two-sided corkscrew condition, along with a Harnack chain condition.

We say that $\Omega \subset \mathbb{R}^n$ satisfies the *interior corkscrew condition* if there are constants $M > 1$ and $R > 0$ such that for each $x \in \partial\Omega$ and $r \in (0,R)$ there exists

$$\begin{aligned} &A_r(x) \in \Omega, \text{ called corkscrew point relative to } x, \\ &\text{so that } |x - A_r(x)| < r \text{ and } \operatorname{dist}(A_r(x),\partial\Omega) > M^{-1}r. \end{aligned} \tag{2.10}$$

Also, $\Omega \subset \mathbb{R}^n$ satisfies the *exterior corkscrew condition* if $\Omega^c := \mathbb{R}^n \setminus \Omega$ satisfy the interior corkscrew condition. Finally, Ω satisfies the *two sided corkscrew condition* if it satisfies both the interior and exterior corkscrew conditions.

The Harnack chain condition is defined as follows (with reference to M and R as above). First, given $x_1, x_2 \in \Omega$, a Harnack chain from x_1 to x_2 in Ω is a sequence of balls $B_1, \dots, B_K \subset \Omega$ such that $x_1 \in B_1$, $x_2 \in B_K$ and $B_j \cap B_{j+1} \neq \emptyset$ for $1 \leq j \leq K-1$, and such that each B_j has a radius r_j satisfying $M^{-1}r_j < \operatorname{dist}(B_j,\partial\Omega) <$

Mr_j. The length of the chain is K. Then the Harnack chain condition on Ω is that if $\varepsilon > 0$ and $x_1, x_2 \in \Omega \cap B_{r/4}(z)$ for some $z \in \partial\Omega$, $r \in (0, R)$, and if $\operatorname{dist}(x_j, \partial\Omega) > \varepsilon$, $j = 1, 2$, and $|x_1 - x_2| < 2^k \varepsilon$, then there exists a Harnack chain $B_1, \ldots, B_K$ from x_1 to x_2, of length $K \leq Mk$, having the further property that the diameter of each ball B_j is $\geq M^{-1} \min\big(\operatorname{dist}(x_1, \partial\Omega), \operatorname{dist}(x_2, \partial\Omega)\big)$.

An open set $\Omega \subset \mathbb{R}^n$ is said to be a *two-sided NTA domain* if both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ are NTA domains.

It is then apparent from Proposition 2.4 and the above definition that the following result holds.

Corollary 2.7. *If $\Omega \subseteq \mathbb{R}^n$ is a domain satisfying a two-sided corkscrew condition and whose boundary is Ahlfors regular, then Ω is a UR domain.*

We now turn to the notion of the non-tangential maximal operator and non-tangential boundary trace of a function in an open set Ω, as defined in (1.1)–(1.3). It should be noted that, without extra conditions on Ω, it could happen that $\Gamma_\kappa(z) = \emptyset$ for points $z \in \partial\Omega$. This point will be discussed further below. Here and elsewhere in the sequel, we make the convention that $\mathcal{N}_\kappa u(z) = 0$ whenever $z \in \partial\Omega$ is such that $\Gamma_\kappa(z) = \emptyset$. For now, we record a result which shows that the choice of κ plays only a relatively minor role when measuring the size of the nontangential maximal function in $L^p(\partial\Omega, d\sigma)$.

Proposition 2.8. *Assume $\Omega \subset \mathbb{R}^n$ is open and Ahlfors regular. Then for every $\kappa_0, \kappa_1 > 0$ and $0 < p < \infty$ there exist $C_0, C_1 > 0$ such that*

$$C_0 \|\mathcal{N}_{\kappa_0} u\|_{L^p(\partial\Omega, d\sigma)} \leq \|\mathcal{N}_{\kappa_1} u\|_{L^p(\partial\Omega, d\sigma)} \leq C_1 \|\mathcal{N}_{\kappa_0} u\|_{L^p(\partial\Omega, d\sigma)}, \tag{2.11}$$

for each function u.

A proof can be found in [13]. For further purposes, let us also record here a useful estimate also proved in [13].

Proposition 2.9. *Assume that $\Omega \subset \mathbb{R}^n$ is an open set with an Ahlfors regular boundary, and fix $\kappa > 0$. Then there exists $C > 0$ depending only on n, κ, and the Ahlfors regularity constant on Ω such that*

$$\frac{1}{\delta} \int_{\mathcal{O}_\delta} |v|\, dx \leq C \|\mathcal{N}_\kappa^\delta v\|_{L^1(\partial\Omega, d\sigma)}, \qquad 0 < \delta \leq \operatorname{diam}(\Omega), \tag{2.12}$$

for any measurable function $v : \Omega \to \mathbb{R}$, where

$$\mathcal{O}_\delta := \{x \in \Omega :\, \operatorname{dist}(x, \partial\Omega) \leq \delta\}, \quad \forall\, \delta > 0, \tag{2.13}$$

and

$$\mathcal{N}_\kappa^\delta v(x) := \sup\{|v(y)| :\, y \in \Gamma_\kappa(x),\ |x - y| \leq 2\delta\}, \tag{2.14}$$

with the convention that $\mathcal{N}_\kappa^\delta v(x) := 0$ whenever the supremum in the right-hand side of (2.14) *is taken over the empty set.*

Generally speaking, given an open set $\Omega \subset \mathbb{R}^n$, $\kappa > 0$ and a function $u : \Omega \to \mathbb{R}$, recall that the nontangential boundary trace of u is considered in the sense of (1.2) whenever the limit exists. For this definition to be pointwise σ-a.e. meaningful, it is necessary that

$$z \in \overline{\Gamma_\kappa(z)} \quad \text{for } \sigma\text{-a.e. } \ z \in \partial\Omega. \tag{2.15}$$

Definition 2.10. An open domain $\Omega \subset \mathbb{R}^n$ is called *weakly accessible* if it satisfies (2.15) above.

This definition has been adopted in [13], where the following result has been established.

Proposition 2.11. *Let $\Omega \subset \mathbb{R}^n$ be an open set with an Ahlfors regular boundary $\partial\Omega$. Assume that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ (so that, in particular, Ω is of locally finite perimeter). Then Ω is a weakly accessible domain.*

The above considerations are relevant in the context of the following version of the Divergence Theorem proved in [13].

Theorem 2.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set which has an Ahlfors regular boundary and which satisfies (2.3). Also, fix $\kappa > 0$. Denote by ν the outward unit normal to $\partial\Omega$ and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Then*

$$\int_\Omega \operatorname{div} \vec{v}\, dx = \int_{\partial\Omega} \langle \nu, \vec{v}\big|_{\partial\Omega} \rangle \, d\sigma \tag{2.16}$$

holds for each vector field $\vec{v} \in C^0(\Omega)$ that satisfies

$$\begin{aligned} &|\vec{v}|, \operatorname{div} \vec{v} \in L^1(\Omega), \quad \mathcal{N}_\kappa \vec{v} \in L^1(\partial\Omega, d\sigma) \text{ and the nontangential} \\ &\text{trace } \ \vec{v}\big|_{\partial\Omega} \ \text{ exists } \ \sigma\text{-a.e. on } \partial\Omega, \text{ in the sense of } (1.2). \end{aligned} \tag{2.17}$$

3. The harmonic measure

We start by briefly recalling the construction of harmonic measure associated with some bounded open set Ω in $\mathbb{R}^n$. Our presentation follows closely that in [14]. Given $f : \partial\Omega \to \mathbb{R}$, define the upper class of functions associated with f as

$$\begin{aligned} U_f \quad := \quad & \{u \equiv \infty \text{ in } \Omega\} \bigcup \{u :\, \Delta u \geq 0 \text{ and } u \geq M > 0 \text{ in } \Omega, \\ & \qquad \liminf_{\Omega \ni x \to y} u(x) \geq f(y) \quad \forall\, y \in \partial\Omega\} \end{aligned} \tag{3.1}$$

and the lower class of functions associated with f as

$$L_f := \{-u :\, u \in U_{-f}\}. \tag{3.2}$$

Furthermore, define

$$\overline{\mathrm{PI}}\, f(x) := \inf\{u(x) :\, u \in U_f\} \quad \text{and} \quad \underline{\mathrm{PI}}\, f(x) := \sup\{u(x) :\, u \in L_f\}, \tag{3.3}$$

i.e., the upper and, respectively, the lower solution of the generalized Dirichlet problem for f. A function f is called *a resolutive boundary function* if $\overline{\mathrm{PI}}\, f(x) =$

$\underline{\mathrm{PI}}\, f(x)$ for all $x \in \Omega$ and $\Delta(\overline{\mathrm{PI}}\, f) = 0$ in Ω. If f is a resolutive boundary function we set $\mathrm{PI}\, f(x) := \overline{\mathrm{PI}}\, f(x)$, for $x \in \Omega$. In [23], Wiener proved that every continuous real-valued function on $\partial\Omega$ is resolutive. This result and the maximum principle for harmonic functions allows the definition of harmonic measure.

Definition 3.1. Let Ω be a bounded domain in $\mathbb{R}^n$ and $x \in \Omega$ be arbitrary. Then the harmonic measure of Ω evaluated at x, denoted by ω^x, is the unique probability Borel measure on $\partial\Omega$ with the property that

$$\mathrm{PI}\, f(x) = \int_{\partial\Omega} f\, d\omega^x, \qquad \forall\, f \in C(\partial\Omega), \quad \forall\, x \in \Omega. \tag{3.4}$$

Definition 3.2. A bounded domain Ω in $\mathbb{R}^n$ is called regular for the Dirichlet problem if, given any $f \in C(\partial\Omega)$ one has $\mathrm{PI}\, f \in C(\overline{\Omega})$ and $\mathrm{PI}\, f(y) = f(y)$ for all $y \in \partial\Omega$.

Necessary and sufficient conditions for regularity are known. To state one of them, recall the following concept of capacity.

Definition 3.3. Assume that $E \subset \mathbb{R}^n$ by a compact set, contained in a ball $B \subset \mathbb{R}^n$ of radius one. Then

$$\mathrm{cap}\,(E) := \inf\Big\{\int_{\mathbb{R}^n} |\nabla\varphi|^2\, dx : \varphi \in C_c^\infty(B), \quad \varphi \geq 1 \text{ on } E\Big\}. \tag{3.5}$$

The following Wiener criterion then holds.

Theorem 3.4. *Assume that Ω is an open, relatively compact subset of $\mathbb{R}^n$. Then Ω is regular if and only if*

$$\int_0^1 \mathrm{cap}\,(\overline{B(x,r)} \setminus \Omega)\, \frac{dr}{r^{n-1}} = +\infty \qquad \textit{for all } x \in \partial\Omega. \tag{3.6}$$

In particular, for a bounded open set $\Omega \subseteq \mathbb{R}^n$,

$$\Omega \textit{ satisfies an exterior corkscrew condition} \Longrightarrow \Omega \textit{ is regular.} \tag{3.7}$$

The basic fact that the harmonic measure in an NTA domain is doubling has been established in [14]. Throughout the paper, we use the notation $\Delta(x,r) := \partial\Omega \cap B(x,r)$, $x \in \partial\Omega$, $r > 0$, to denote surface balls.

Lemma 3.5. *Let Ω be a bounded NTA domain in $\mathbb{R}^n$, and assume that $x_o \in \Omega$ is fixed. Then the harmonic measure ω^{x_o} is doubling. That is, there exists a finite constant $C = C(\Omega, x_o) > 0$ with the property that*

$$\omega^{x_o}(\Delta(x,2r)) \leq C\omega^{x_o}(\Delta(x,r)), \qquad x \in \partial\Omega, \ r > 0. \tag{3.8}$$

Next, recall the concept of corkscrew point from (2.10). The following is a generalization of a similar result proved by B. Dahlberg in Lipschitz domains in [4]. It appears as Lemma 4.8 on p. 86 in [14].

Lemma 3.6. *If Ω is a bounded NTA domain in $\mathbb{R}^n$, then there exists $R > 0$ with the following property. Assume that $x_0, x_1 \in \partial\Omega$ and $0 < r_0, r_1 < R$ are such that $\Delta(x_1, r_1) \subset \Delta(x_0, r_0/2)$. Then*

$$\omega^{A_r(x_0)}(\Delta(x_1, r_1)) \approx \frac{\omega^x(\Delta(x_1, r_1))}{\omega^x(\Delta(x_0, r_0))}, \tag{3.9}$$

uniformly for $x \in \Omega \setminus B(x_0, 2r_0)$.

Here and elsewhere, $C_1 \approx C_2$ means that C_1/C_2 is bounded above and below by constants which depend only of the NTA character of Ω.

As remarked in [14], if x_1, x_2 are two arbitrary points in Ω, then the measures ω^{x_1} and ω^{x_2} are mutually absolutely continuous. This can be seen by using Harnak's inequality. As such, the Radon-Nikodym derivative $d\omega^{x_1}/d\omega^{x_2}$ exists and, hence, the following definition is meaningful.

Definition 3.7. Let Ω be a bounded NTA domain in $\mathbb{R}^n$ and fix $x_0 \in \Omega$. Then the kernel function associated with Ω is defined as the Radon-Nikodym derivative

$$K(x, \cdot) := \frac{d\omega^x}{d\omega^{x_0}} \quad \text{on} \ \ \partial\Omega. \tag{3.10}$$

In particular, if $x_0 \in \Omega$ is fixed, then for every $x \in \Omega$ one has

$$K(x, y) = \lim_{\Delta' \searrow y} \frac{\omega^x(\Delta')}{\omega^{x_0}(\Delta')}, \qquad \Delta' \ \ \text{surface ball}, \tag{3.11}$$

for ω^{x_0}-a.e. $y \in \partial\Omega$. Even though, a priori, $K(x, y)$ is only defined for ω^{x_0}-a.e. $y \in \partial\Omega$, it actually turns out that $K(x, y)$ is a Hölder continuous function in the y variable (cf. [14]). We continue by recording several estimates on the kernel function which are due to D. Jerison and C. Kenig.

Lemma 3.8. *Let Ω be a bounded NTA domain in $\mathbb{R}^n$, and fix $x_0 \in \Omega$ and $y_0 \in \partial\Omega$. Then there exists $r_0 > 0$ such that*

$$K(A_r(y_0), y) \approx 1/(\omega^{x_0}(\Delta(y, r))), \ \ \text{uniformly for } y \in \partial\Omega \setminus B(y_0, r),\ 0 < r < r_0. \tag{3.12}$$

Indeed, estimate (3.12) is a consequence of (3.11) and Lemma 3.6.

We continue with a review of conditions guaranteeing that the harmonic measure of a domain is absolutely continuous with respect to its surface measure.

Proposition 3.9. *Let $\Omega \subset \mathbb{R}^n$ be an open set, with an Ahlfors regular boundary $\partial\Omega$ which satisfies the "two disks" condition introduced in Proposition 2.4. In addition, assume that Ω satisfies an interior corkscrew condition and the Harnack chain condition (cf. Definition 2.6). Fix $x_o \in \Omega$ and denote by ω^{x_o} the harmonic measure on $\partial\Omega$ (relative to Ω) with pole at x_o.*

Then ω^{x_o} belongs to the Muckenhoupt class A_∞ with respect to $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. In particular, ω^{x_o} and σ are mutually absolutely continuous.

With the two disks condition replaced by a two-sided corkscrew condition and when $\mathbb{R}^n \setminus \partial\Omega$ has precisely two connected components, this has been first obtained in [20]. In the current format, the above result appears as Theorem 2 on p. 842 in [6]. A result of a similar flavor, when the Harnack chain condition is suppressed, has been established by B. Bennewitz and J. Lewis in [1]. Their Theorem 1 (where a *weak* reverse Hölder estimate is proved) entails the following:

Proposition 3.10. *In the context of Proposition* 3.9*, the mutual absolute continuity of* ω^{x_o} *and* σ *remains valid even when the assumption that* Ω *satisfies the Harnack chain condition is dropped.*

The following Fatou type theorem will play a major role in future considerations.

Proposition 3.11. *Assume that* $\Omega \subset \mathbb{R}^n$ *is an NTA domain with an Ahlfors regular boundary and fix* $\kappa > 0$*. Then for every* $0 < p \leq \infty$*,*

$$\Delta u = 0 \text{ in } \Omega \text{ and } \mathcal{N}_\kappa u \in L^p(\partial\Omega, d\sigma) \implies u\big|_{\partial\Omega} \text{ exists } \sigma\text{-a.e.} \tag{3.13}$$

in the nontangential pointwise sense (*cf.* (1.2))*.*

Proof. Fix $x_0 \in \Omega$. As a consequence of [14, Theorem 6, p. 112], if u is harmonic in Ω and $\|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega, d\sigma)} < \infty$, then (1.2) exists for ω^{x_0}-a.e. $z \in \partial\Omega$. Then (3.13) follows from this and Proposition 3.9. □

We conclude this section by recording Theorem 5.8 from p. 105 in [14], which provides a way of extending boundary functions (which are absolutely integrable with respect to the harmonic measure) harmonically inside of an NTA domain.

Theorem 3.12. *Suppose* Ω *is an NTA domain in* $\mathbb{R}^n$ *and that* $x_o \in \Omega$, $\kappa > 0$ *are fixed. Also, recall the kernel function* $K(\cdot, \cdot)$ *from* (3.10)*. If for* $f \in L^1(\partial\Omega, d\omega^{x_o})$ *one defines*

$$u(x) := \int_{\partial\Omega} K(x, y) f(y)\, d\omega^{x_o}(y), \qquad x \in \Omega, \tag{3.14}$$

then u *is a well-defined function, which is harmonic in* Ω *and satisfies*

$$u\big|_{\partial\Omega} = f \quad \text{at } \omega^{x_o}\text{-a.e. point on } \partial\Omega, \text{ and} \tag{3.15}$$

$$(\mathcal{N}_\kappa u)(y) \approx (\mathcal{M}_{\omega^{x_o}} f)(y) \quad \text{uniformly for } y \in \partial\Omega, \tag{3.16}$$

where $\mathcal{M}_{\omega^{x_o}}$ *is the Hardy-Littlewood maximal function on* $\partial\Omega$ *associated with the harmonic measure* ω^{x_0}*.*

Strictly speaking, Theorem 5.8 from p. 105 in [14] only contains the left-pointing inequality in (3.16), but the opposite one is proved much as in Lemma 1.4.2 on p. 14 in [15]. More specifically, let $f \in L^1(\partial\Omega, d\omega^{x_0})$, $f \geq 0$, and let r_0 be as in Lemma 3.8. Then for $z \in \partial\Omega$, $0 < r < r_0$ and $x \in [B(z, 2r) \setminus B(z, r)] \cap \Gamma_\kappa(z)$, we may write

$$⨍_{\Delta(z,r)} f(y)\, d\omega^{x_0}(y) \leq C \int_{\partial\Omega} K(x, y) f(y)\, d\omega^{x_0}(y) \leq C u(x) \leq C(\mathcal{N}_\kappa u)(z), \tag{3.17}$$

by (3.12). This readily yields the desired result.

4. The Green function

Let Ω be an open set in $\mathbb{R}^n$. By $W^{k,p}(\Omega)$, $1 \le p \le \infty$, $k \in \mathbb{Z}$, we shall denote the classical L^p-based Sobolev space of order k in Ω. Also, we let $\mathring{W}^{k,p}(\Omega)$ denote the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. Going further, for a measure space (X,μ) and $0 < p < \infty$, the weak L^p-Lebesgue spaces are defined as

$$L^{p,\infty}(X) := \Big\{ f : X \to \mathbb{R} \text{ measurable} : \sup_{\lambda > 0} \Big(\lambda^p \mu(\{x \in X : |f(x)| > \lambda\}) \Big) < \infty \Big\},$$

equipped with

$$\|f\|_{L^{p,\infty}(X)} := \sup_{\lambda>0} \Big(\lambda\, \mu(\{x \in X : |f(x)| > \lambda\})^{1/p} \Big). \tag{4.1}$$

The Green function for the Laplacian in arbitrary open sets is defined in the theorem below, where we also collect its most basic properties in such a setting (for proofs, we refer to [12]). In two subsequent theorems we then indicate how some of these properties improve as the underlying domain becomes more regular.

Theorem 4.1. *Assume that Ω is an arbitrary, bounded open set in $\mathbb{R}^n$. Then there exists a unique function $G : \Omega \times \Omega \to [0, +\infty]$ such that*

$$G(\cdot, y) \in W^{1,2}(\Omega \setminus B(y,r)) \cap \mathring{W}^{1,1}(\Omega), \qquad \forall\, y \in \Omega,\ \ \forall\, r > 0, \tag{4.2}$$

and

$$\int_\Omega \langle \nabla_x G(x,y), \nabla \varphi(x) \rangle \, dx = \varphi(y), \quad \forall\, \varphi \in C_c^\infty(\Omega). \tag{4.3}$$

Furthermore, the Green function also satisfies the following additional properties:

(i) $G(x,y) = G(y,x)$, *for all* $x, y \in \Omega$;
(ii) $G(x,y) \le C_n |x-y|^{2-n}$, *for all* $x, y \in \Omega$;
(iii) $G(x,y) \ge C_n |x-y|^{2-n}$, *for all* $x, y \in \Omega$ *with* $|x-y| \le \frac{1}{2} \operatorname{dist}(y, \partial\Omega)$;
(iv) $G(\cdot, y) \in \mathring{W}^{1,p}(\Omega)$ *for all* $1 \le p < \frac{n}{n-1}$, *uniformly in* $y \in \Omega$;
(v) $G(\cdot, y) \in L^{\frac{n}{n-2},\infty}(\Omega)$, *uniformly in* $y \in \Omega$;
(vi) $\nabla G(\cdot, y) \in L^{\frac{n}{n-1},\infty}(\Omega)$, *uniformly in* $y \in \Omega$.

From (4.2)–(4.3) we see that

$$-\Delta G(\cdot, y) = \delta_y \ \text{ for each fixed } y \in \Omega, \tag{4.4}$$

so that also

$$-\Delta G(x, \cdot) = \delta_x \ \text{ for each fixed } x \in \Omega, \tag{4.5}$$

by (i) above. For further use, let us also remark that if $y \in \Omega$ is fixed, then for every $\varphi \in C_c^\infty(\Omega)$ we have, on account of (4.2)–(4.3),

$$\begin{aligned}\varphi(y) &= \int_\Omega \langle \nabla_x G(x,y), \nabla\varphi(x)\rangle\, dx = \lim_{\varepsilon\to 0^+} \int_{\Omega\setminus\overline{B(y,\varepsilon)}} \langle \nabla_x G(x,y), \nabla\varphi(x)\rangle\, dx \\ &= -\lim_{\varepsilon\to 0^+} \int_{\partial B(y,\varepsilon)} \partial_{\nu(x)} G(x,y)\varphi(x)\,\mathcal{H}^{n-1}(x), \qquad (4.6)\end{aligned}$$

given that φ vanishes near $\partial\Omega$ and $G(\cdot,y)$ is harmonic in a neighborhood of $(\operatorname{supp}\varphi)\setminus B(y,\varepsilon)$. Consequently,

$$\psi(y) = -\lim_{\varepsilon\to 0^+} \int_{\partial B(y,\varepsilon)} \partial_{\nu(x)} G(x,y)\psi(x)\,\mathcal{H}^{n-1}(x), \quad \forall\, y\in\Omega,\ \forall \psi \in C^\infty \text{ near } y. \qquad (4.7)$$

Before stating the next theorem we make, following [15], a definition.

Definition 4.2. A bounded open set $\Omega \subset \mathbb{R}^n$ is called of class S if there exist $\alpha \in (0,1]$ and $r_0 > 0$ such that

$$\mathcal{H}^n(B(x_0,r)\cap\Omega^c) \geq \alpha r^n \qquad (4.8)$$

for all $x_0 \in \Omega^c$, $0 < r \leq r_0$.

The result below is contained in [15, Theorem 1.2.8 on p. 7].

Theorem 4.3. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set of class S. Then, for some (typically small) number $\alpha = \alpha(\Omega) > 0$, the Green function associated with Ω also satisfies:*

- (i) $G(x,y) \leq C \operatorname{dist}(y,\partial\Omega)^\alpha |x-y|^{2-n-\alpha}$ *for all* $x,y\in\Omega$;
- (ii) $|G(x,y)-G(z,y)| \leq C|x-z|^\alpha/(|x-y|^{2-n-\alpha}+|z-y|^{2-n-\alpha})$ *for all* $x,y,z\in\Omega$.

In particular, if $\Omega \subset \mathbb{R}^n$ is a bounded open set of class S, then, for every fixed $y \in \Omega$,

$$G(\cdot,y) \text{ is Hölder continuous near } \partial\Omega \text{ and } G(\cdot,y)\big|_{\partial\Omega} = 0. \qquad (4.9)$$

We continue by recording the following definition.

Definition 4.4. An open set $\Omega \subset \mathbb{R}^n$ is said to satisfy a uniform exterior ball condition (henceforth abbreviated as UEBC), if there exists $r > 0$ with the following property: For each $x \in \partial\Omega$, there exists a point $y = y(x) \in \mathbb{R}^n$ such that

$$\overline{B(y,r)} \setminus \{x\} \subseteq \mathbb{R}^n \setminus \Omega \quad\text{and}\quad x \in \partial B(y,r). \qquad (4.10)$$

The largest radius r satisfying the above property will be referred to as the UEBC constant of Ω.

For domains satisfying a UEBC, the behavior of the Green function further improves as indicated in the following theorem due to M. Grüter and K.-O. Widman (see Theorem 3.3 in [12]).

Theorem 4.5. *If $\Omega \subset \mathbb{R}^n$ is a bounded open set which satisfies a UEBC, then its associated Green function also satisfies the following five properties for all $x, y \in \Omega$:*

(i) $G(x,y) \leq C \operatorname{dist}(x, \partial\Omega)|x-y|^{1-n}$;
(ii) $G(x,y) \leq C \operatorname{dist}(x, \partial\Omega) \operatorname{dist}(y, \partial\Omega)|x-y|^{-n}$;
(iii) $|\nabla_x G(x,y)| \leq C|x-y|^{1-n}$;
(iv) $|\nabla_x G(x,y)| \leq C \operatorname{dist}(y, \partial\Omega)|x-y|^{-n}$;
(v) $|\nabla_x \nabla_y G(x,y)| \leq C|x-y|^{-n}$.

We continue by recording Lemma 4.8 from [14].

Lemma 4.6. *Let Ω be a bounded NTA domain in $\mathbb{R}^n$. Then there exist constants $r_0 > 0$, $M > 0$ which depend only on Ω such that if $0 < r < r_0/2$, $y \in \partial\Omega$, and $x \in \Omega \setminus B(y, 2r)$, then*

$$M^{-1} < \frac{\omega^x(\Delta(y,r))}{r^{n-2}|G(A_r(y), x)|} < M, \tag{4.11}$$

where $G(x,y)$ is the Green function of Ω and $A_r(y) \in \Omega$ is a corkscrew point (relative to y, at scale r; cf. (2.10)*), i.e.,*

$$M^{-1}r < |A_r(y) - y| < Mr \quad \text{and} \quad \operatorname{dist}(A_r(y), \partial\Omega) > M^{-1}r. \tag{4.12}$$

The last result in this section is a version of Theorem 1.7.3 on p. 29 in [15] (strictly speaking, the latter was stated when the underlying domain is a ball in $\mathbb{R}^n$ but, as remarked at the beginning of [15, Chapter 1, Section 3], essentially the same proof works for the class of NTA domains).

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain with Ahlfors regular boundary. Denote by ω^{x_0} the harmonic measure with pole at a fixed point $x_0 \in \Omega$, and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix some $\kappa > 0$. Then the following statements are equivalent:*

(i) *There holds $\omega^{x_0} \in A_\infty(d\sigma)$;*
(ii) *There exist $1 < p < \infty$ and $C = C(\Omega, p, \kappa) > 0$ such that if $f \in C(\partial\Omega)$ and u is the solution of the classical Dirichlet problem*

$$\begin{cases} \Delta u = 0 \ \text{ in } \Omega, \\ u \in C(\overline{\Omega}), \\ u\big|_{\partial\Omega} = f, \end{cases} \tag{4.13}$$

then

$$\|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega, d\sigma)} \leq C\|f\|_{L^p(\partial\Omega, d\sigma)}; \tag{4.14}$$

(iii) *The harmonic measure ω^{x_0} is absolutely continuous with respect to σ and $k := \frac{d\omega^{x_0}}{d\sigma}$ belongs to the reverse Hölder class $B_{p'}(d\sigma)$, for $\frac{1}{p} + \frac{1}{p'} = 1$ (that is, $\Big(⨍_\Delta k^{p'}\, d\sigma\Big)^{1/p'} \leq ⨍_\Delta k\, d\sigma$ for all surface balls $\Delta \subset \partial\Omega$).*

5. Proofs of main results

In Theorem 5.1 and Theorem 5.3 below we deal, respectively, with the existence and uniqueness of a solution for the Dirichlet problem.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain with an Ahlfors regular boundary, and which satisfies a UEBC. Denote by ν the outward unit normal to Ω and let $G(\cdot,\cdot)$ stand for the Green function associated with Ω. Fix $\kappa > 0$, $1 < p < \infty$, and $f \in L^p(\partial\Omega)$. Then the function u defined as in* (1.6) *solves the Dirichlet problem* (1.4) *and satisfies* (1.5)*.*

Proof. Granted that $\Omega \subset \mathbb{R}^n$ is a domain which satisfies a UEBC, the Green function satisfies (1.11), where C depends only on n and the UEBC constant of Ω. See (iv) in Theorem 4.5. Then, for each $y \in \Omega$ and each $0 < r < \operatorname{dist}(y,\partial\Omega)/2$ fixed, $\Omega \setminus B(y,r) \ni x \mapsto \nabla_x G(x,y) \in \mathbb{R}^n$ is a vector field whose components are harmonic and, by (1.11), also bounded. Thus, by Proposition 3.11, we conclude that, for each fixed $y \in \Omega$,

$$(\nabla_x G(x,y))\Big|_{x\in\partial\Omega} \quad \text{exists for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.1}$$

where the boundary trace is taken in the sense of (1.2). Together, estimate (iv) in Theorem 4.5 and (5.1) then imply, by allowing x to approach nontangential boundary points, that for each fixed $y \in \Omega$,

$$|\nabla_x G(x,y)| \le C \operatorname{dist}(y,\partial\Omega)|x-y|^{-n}, \qquad \text{for a.e. } x \in \partial\Omega. \tag{5.2}$$

Note that this further entails

$$-\partial_{\nu(\cdot)} G(\cdot,y) \in L^\infty(\partial\Omega, d\sigma), \qquad \forall\, y \in \Omega. \tag{5.3}$$

As a consequence, if $1 < p < \infty$ and $f \in L^p(\partial\Omega, d\sigma)$ is fixed, then u in (1.6) is a well-defined, harmonic function in Ω (for the latter claim see (4.5)). At the moment, our goal is to show that $\mathcal{N}_\kappa u \in L^p(\partial\Omega, d\sigma)$, where $\kappa > 0$ is fixed. To this end, we remark that from (5.2) and (1.6) we have

$$|u(y)| \le C \int_{\partial\Omega} \frac{\operatorname{dist}(y,\partial\Omega)}{|x-y|^n}\, |f(x)|\, d\sigma(x), \qquad \forall\, y \in \Omega. \tag{5.4}$$

Next, consider an arbitrary boundary point $y_0 \in \partial\Omega$. We claim that there exists a constant $C = C(\partial\Omega,\kappa) > 0$ such that

$$|x-y| \ge C(\operatorname{dist}(y,\partial\Omega) + |x-y_0|), \quad \forall\, y \in \Gamma_\kappa(y_0),\ x \in \partial\Omega. \tag{5.5}$$

Indeed, the fact that $|x-y| \ge \operatorname{dist}(y,\partial\Omega)$ is immediate. Then, if $z \in \partial\Gamma_\kappa(y_0) \subset \overline{\Omega}$ is such that $\operatorname{dist}(x,\Gamma_\kappa(y_0)) = |x-z|$, we have that $|y_0 - z| = (1+\kappa)\operatorname{dist}(z,\partial\Omega)$. Hence we can write

$$\begin{aligned} |x-y_0| &\le |x-z| + |z-y_0| = |x-z| + (1+\kappa)\operatorname{dist}(z,\partial\Omega) \\ &\le (2+\kappa)|x-z|. \end{aligned} \tag{5.6}$$

Thus, $|x-y_0| \le (2+\kappa)\,\mathrm{dist}\,(x,\Gamma_\kappa(y_0)) \le (2+\kappa)\,|x-y|$, and (5.5) follows. Making use of (5.5) in (5.4) we therefore arrive at

$$|u(y)| \le C \int_{\partial\Omega} \frac{\mathrm{dist}\,(y,\partial\Omega)}{(\mathrm{dist}\,(y,\partial\Omega)+|x-y_0|)^n}|f(x)|\,d\sigma(x), \quad \forall\, y \in \Gamma_\kappa(y_0). \tag{5.7}$$

Let us now fix $y \in \Gamma_\kappa(y_0)$ and define $r := \mathrm{dist}\,(y,\partial\Omega)$. In this setting, we use a familiar argument based on decomposing $\partial\Omega$ into a family of dyadic annuli $\partial\Omega = \bigcup_{j=0}^N R_j(y_0)$, where $R_0(y_0) = \Delta(y_0,r)$ and $R_j(y_0) := \Delta(y_0, 2^{j+1}r) \setminus \Delta(y_0, 2^j r)$ for $0 \le j \le N-1$. Using that $\partial\Omega$ is Ahlfors regular, we can then estimate

$$\begin{aligned} \int_{R_j(y_0)} \frac{r}{(r+|x-y_0|)^n}|f(x)|\,d\sigma(x) \quad &\le \quad \frac{C}{r^{n-1}2^{jn}} \int_{\Delta(y_0,2^{j+1}r)} |f|\,d\sigma \\ &\le \quad C2^{-j}\,\mathcal{M}_\sigma f(y_0), \end{aligned} \tag{5.8}$$

uniformly in $j \ge 1$, where $\mathcal{M}_\sigma$ denotes the Hardy-Littlewood maximal function with respect to the surface measure on $\partial\Omega$ (cf. (2.7)). Also,

$$\begin{aligned} \int_{R_0(y_0)} \frac{r}{(r+|x-y_0|)^n}|f(x)|\,d\sigma(x) \quad &\le \quad \frac{C}{r^{n-1}} \int_{\Delta(y_0,r)} |f|\,d\sigma \\ &\le \quad C\mathcal{M}_\sigma f(y_0), \end{aligned} \tag{5.9}$$

thus, on account of (5.7)–(5.9), we obtain that

$$(\mathcal{N}_\kappa u)(y_0) \le C(\mathcal{M}_\sigma f)(y_0), \qquad \forall\, y_0 \in \partial\Omega, \tag{5.10}$$

where $C = C(\Omega,\kappa,p) > 0$ is a finite constant. Hence, for every $1 < p < \infty$,

$$\|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,d\sigma)} \le C(\Omega,\kappa,p)\|f\|_{L^p(\partial\Omega,d\sigma)}, \tag{5.11}$$

by the boundedness of $\mathcal{M}_\sigma$ on $L^p(\partial\Omega,d\sigma)$. Let us also remark that, thanks to (5.11) and Proposition 3.11,

$$\begin{aligned} &u\big|_{\partial\Omega} \text{ exists, belongs to } L^p(\partial\Omega,d\sigma) \\ &\text{and } \big\|u\big|_{\partial\Omega}\big\|_{L^p(\partial\Omega,d\sigma)} \le C(\Omega,p)\|f\|_{L^p(\partial\Omega,d\sigma)}. \end{aligned} \tag{5.12}$$

In summary, in order to conclude that u defined as in (1.6) is a solution of (1.4) which satisfies (1.5), we are left with proving that its nontangential boundary trace matches the given datum f for σ-a.e. point on $\partial\Omega$. With this in mind, consider the linear assignment $T : L^p(\partial\Omega,d\sigma) \to L^p(\partial\Omega,d\sigma)$, given by

$$L^p(\partial\Omega,d\sigma) \ni f \mapsto Tf := u\big|_{\partial\Omega} \in L^p(\partial\Omega,d\sigma), \tag{5.13}$$

with u as in (1.6). Thus, T is well defined, linear and bounded, thanks to (5.12), so it suffices to show that $Tf = f$ for f belonging to a dense subset of $L^p(\partial\Omega,d\sigma)$. In this regard, we make the claim that if $1 < p < \infty$, then

$$\big\{\psi\big|_{\partial\Omega} \colon \psi \in C_c^\infty(\mathbb{R}^n)\big\} \hookrightarrow L^p(\partial\Omega,d\sigma) \qquad \text{densely}. \tag{5.14}$$

In light of Lemma 5.2, and given that any Lipschitz function defined on the compact set $\partial\Omega$ can be extended to a compactly supported Lipschitz function in $\mathbb{R}^n$, (5.14) follows by observing that any such function can, in turn, be approximated uniformly on compact sets by functions from $C_c^\infty(\mathbb{R}^n)$. Granted these considerations, we are therefore left with checking that if $\psi \in C_c^\infty(\mathbb{R}^n)$ then

$$\Big(-\int_{\partial\Omega} \partial_{\nu(x)} G(x,y)\psi(x)\,d\sigma(x)\Big)\Big|_{y\in\partial\Omega} = \psi(y), \qquad \text{for } \sigma\text{-a.e. } y \in \partial\Omega. \tag{5.15}$$

To justify formula (5.15), fix a function $\psi \in C_c^\infty(\mathbb{R}^n)$ along with an arbitrary point $y \in \Omega$, and pick an arbitrary $\varepsilon > 0$. We then make use of Theorem 2.12 in the context in which the vector field v is given by $v(x) := -\psi(x)\nabla_x G(x,y) + G(x,y)(\nabla\psi)(x)$ for $x \in \Omega \setminus \overline{B(y,\varepsilon)}$. Note that, in this case,

$$\operatorname{div} v(x) = G(x,y)\Delta\psi(x), \qquad x \in \Omega \setminus \overline{B(y,\varepsilon)}. \tag{5.16}$$

Upon recalling (5.1), (4.9) (clearly, any domain satisfying an exterior corkscrew condition is of class S), and since $\mathcal{N}[x \mapsto \nabla_x G(x,y)] \in L^\infty(\partial\Omega, d\sigma)$ by (1.11), it follows that the hypotheses of Theorem 2.12 are satisfied. Hence, keeping (4.9) in mind, we may write

$$\begin{aligned}\int_{\Omega\setminus\overline{B(y,\varepsilon)}} G(x,y)(\Delta\psi)(x)\,dx &= -\int_{\partial\Omega} \partial_{\nu(x)} G(x,y)\psi(x)\,d\sigma(x) \\ &\quad + \int_{\partial B(y,\varepsilon)} \big[\psi(x)\partial_{\nu(x)} G(x,y) - (\partial_\nu \psi)(x) G(x,y)\big]\,\mathcal{H}^{n-1}(x).\end{aligned} \tag{5.17}$$

Given (ii) in Theorem 4.1 and (4.7), by passing to limit $\varepsilon \to 0^+$ in the above formula we finally arrive at the conclusion that

$$-\int_{\partial\Omega} \partial_{\nu(x)} G(x,y)\psi(x)\,d\sigma(x) = \psi(y) + \int_\Omega G(x,y)(\Delta\psi)(x)\,dx, \ \ \forall\, y \in \Omega. \tag{5.18}$$

Thus, in order to finish the proof of (5.15), there remains to show that if $\psi \in C_c^\infty(\mathbb{R}^n)$ then

$$\Big(\int_\Omega G(x,y)(\Delta\psi)(x)\,dx\Big)\Big|_{y\in\partial\Omega} = 0, \qquad \text{for } \sigma\text{-a.e. } y \in \partial\Omega. \tag{5.19}$$

To see that this indeed is the case, fix an arbitrary point $y_* \in \partial\Omega$, and select a sequence $y_j \in \Omega$, $j \in \mathbb{N}$, such that $\lim_{j\to\infty} y_j = y_*$. For each $j \in \mathbb{N}$, set $F_j(x) := G(x, y_j)(\Delta\psi)(x)$ with $x \in \Omega$. Then, for each fixed $x \in \Omega$, we have that $\lim_{j\to\infty} F_j(x) = 0$ by (4.9) and (i) in Theorem 4.1. Moreover, (ii) in Theorem 4.1 can be used to readily justify the fact that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable set $E \subseteq \Omega$,

$$\mathcal{H}^n(E) < \delta \Longrightarrow \int_E |F_j(x)|\,dx < \varepsilon \ \text{ for every } j \in \mathbb{N}. \tag{5.20}$$

In other words, the family of functions $\{F_j\}_{j\in\mathbb{N}}$ has uniformly absolutely continuous integrals and, since $\mathcal{H}^n(\Omega) < +\infty$, Vitali's theorem applies and yields $\lim_{j\to\infty} \int_\Omega F_j(x)\,dx = 0$. Hence, (5.19) holds, and this finishes the proof of (5.15).

Altogether, the above reasoning shows that, given $f \in L^p(\partial\Omega, d\sigma)$, where $1 < p < \infty$, the function u given by (1.6) solves (1.4) and satisfies (1.5). □

Here is the density lemma (see [13]) which has been used in the course of the above proof.

Lemma 5.2. *Assume that Σ is a locally compact metric space and that σ is a locally finite Borel measure on Σ. Also, denote by* $\mathrm{Lip}_o(\Sigma)$ *the space of compactly supported, Lipschitz functions on Σ. Then for every $p \in [1,\infty)$ the inclusion*

$$\mathrm{Lip}_o(\Sigma) \hookrightarrow L^p(\Sigma, d\sigma) \tag{5.21}$$

has dense range.

Our next result deals with the issue of uniqueness for $(D)_p$. In contrast to Theorem 5.1, note that the (bounded) domain in question is merely NTA, with an Ahlfors regular boundary.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain with an Ahlfors regular boundary. Assume that for some fixed $p \in (1,\infty)$, a solution u to the problem* (1.4) (*with $\kappa > 0$ fixed*) *can be found which satisfies* (1.5). *Then, necessarily, this solution is also unique.*

Proof. Our argument combines ideas from [13] and [15, Theorem 1.7.7, p. 30]. As a preamble, we first note the following useful consequence of the current working assumptions and Theorem 4.7. Specifically, if $x_0 \in \Omega$ is fixed and p' is the conjugate of p, then ω^{x_0} is absolutely continuous with respect to σ and $k := \frac{d\omega^{x_0}}{d\sigma} \in B_{p'}(d\sigma)$, the reverse Hölder class (see Section 6). That is,

$$k \in L^{p'}(\partial\Omega, d\sigma) \quad \text{and} \quad \Big(\fint_{\Delta} k^{p'}\,d\sigma\Big)^{1/p'} \leq \fint_{\Delta} k\,d\sigma, \tag{5.22}$$

for all surface balls $\Delta \subseteq \partial\Omega$. In particular, if we introduce

$$m(z) := \sup_{\Delta \ni Z} \frac{\omega^{x_0}(\Delta)}{\sigma(\Delta)} = \sup_{\Delta \ni Z} \fint_{\Delta} k\,d\sigma = \mathcal{M}_\sigma k(z), \qquad z \in \partial\Omega, \tag{5.23}$$

then

$$\Big(\int_{\partial\Omega} m(z)^{p'}\,d\sigma(z)\Big)^{1/p'} \leq C\|k\|_{L^{p'}(\partial\Omega, d\sigma)} < +\infty, \tag{5.24}$$

by the boundedness of the Hardy-Littlewood maximal function on $L^{p'}(\partial\Omega, d\sigma)$.

Continuing to lay the ground for subsequent arguments, we consider the (one-sided) δ-collar of the boundary, i.e., $\mathcal{O}_\delta := \{z \in \Omega :\ \mathrm{dist}\,(z, \partial\Omega) \leq \delta\}$ where $0 < \delta < \mathrm{diam}\,(\Omega)$, and pick a family of functions ψ_δ, indexed by δ, where $0 < \delta < \frac{1}{4}\,\mathrm{dist}\,(x_0, \partial\Omega)$, with the following properties:

$$\psi_\delta \in C^\infty_c(\Omega), \quad 0 \leq \psi_\delta \leq 1, \quad |\partial^\alpha \psi_\delta| \leq C_\alpha \delta^{-|\alpha|} \quad \forall\,\alpha, \tag{5.25}$$

$$\psi_\delta \equiv 1 \text{ on } \Omega \setminus \mathcal{O}_\delta \quad \text{and} \quad \psi_\delta \equiv 0 \text{ on } \mathcal{O}_{\delta/2}. \tag{5.26}$$

Such a family can be constructed by starting with some $\eta \in C^\infty(\mathbb{R})$ with the property that $\eta \equiv 0$ on $(-\infty, c_1)$ and $\eta \equiv 1$ on $(c_2, +\infty)$, where $0 < c_1 < c_2 < \infty$

are some suitably chosen constants. Then, if $\rho_{\rm reg}$ denotes the regularized distance to $\mathbb{R}^n \setminus \Omega$ (in the sense of [21, Theorem 2, p. 171]), we may take

$$\psi_\delta(x) := \psi(\delta^{-1}\rho_{\rm reg}(x)), \qquad x \in \mathbb{R}^n. \tag{5.27}$$

Conditions (5.25)–(5.26) now follow from

$$\begin{aligned} &C_1 \,{\rm dist}\,(x,\partial\Omega) \leq \rho_{\rm reg}(x) \leq C_2 \,{\rm dist}\,(x,\partial\Omega), \\ &|\partial^\alpha \rho_{\rm reg}(x)| \leq C_\alpha \,{\rm dist}\,(x,\partial\Omega)^{1-|\alpha|}. \end{aligned} \tag{5.28}$$

Turning to the uniqueness part in earnest, denote by $G(\cdot,\cdot)$ the Green function associated with Ω. Also, we let u be a solution of $(D)_p$ with $f=0$. Then, if $0<\delta<\frac{1}{4}\,{\rm dist}\,(x_0,\partial\Omega)$, $\psi_\delta u \in C^\infty_c(\Omega)$ and successive integrations by parts give

$$\begin{aligned} -u(x_0) = -(\psi_\delta u)(x_0) &= -\int_\Omega \langle \nabla_y G(x_0,y), \nabla_y(\psi_\delta u)(y)\rangle\,dy \\ &= \int_\Omega G(x_0,y)\Delta_y(\psi_\delta u)(y)\,dy \\ &= 2\int_\Omega G(x_0,y)\langle \nabla\psi_\delta(y), \nabla u(y)\rangle\,dy + \int_\Omega G(x_0,y)(\Delta\psi_\delta)(y)u(y)\,dy \\ &= -2\int_\Omega \langle \nabla_y G(x_0,y), (\nabla\psi_\delta)(y)\rangle u(y)\,dy - \int_\Omega G(x_0,y)(\Delta\psi_\delta)(y)u(y)\,dy, \\ &=: I + II, \end{aligned} \tag{5.29}$$

since $\Delta u = 0$ in Ω and $\psi_\delta \equiv 0$ near $\partial\Omega$. In order to be able to estimate I in (5.29) we need some preparations. To this end, let $\{I_k\}_k$ be a decomposition of Ω into nonoverlapping Whitney cubes and, for each fixed $\delta>0$, set

$$\mathcal{J}_\delta := \{k:\ I^\delta_k := I_k \cap \mathcal{O}_\delta \neq \emptyset\}. \tag{5.30}$$

It follows that the side-length of each I^δ_k is comparable with δ. Going further, since $\partial\Omega$ (equipped with the measure σ and the Euclidean distance) is a space of homogeneous type, there exists a decomposition of $\partial\Omega$ into a grid of dyadic boundary "cubes" Q^δ, of side-length comparable with δ. For each $k \in \mathcal{J}_\delta$, select one such boundary dyadic cube Q^δ_k with the property that

$${\rm dist}\,(I^\delta_k,\partial\Omega) = {\rm dist}\,(I^\delta_k, Q^\delta_k). \tag{5.31}$$

Matters can be arranged so that the concentric dilates of these boundary dyadic cubes have bounded overlap. That is, for every $c \geq 1$ there exists a finite constant $C>0$ such that

$$\sum_{k\in\mathcal{J}_\delta} \mathbf{1}_{c\,Q^\delta_k} \leq C \quad \text{on} \quad \partial\Omega. \tag{5.32}$$

Next, assume that $\delta>0$ is much smaller than ${\rm dist}\,(x_0,\partial\Omega)$. Recall (5.23) and note that every point $y \in I^\delta_k$ is a corkscrew point, relative to any point $z \in Q^\delta_k$, at scale δ. Consequently, (4.11) gives that

$$\begin{aligned} m(z) &\geq \frac{\omega^{x_0}(\Delta(z,\delta))}{\sigma(\Delta(z,\delta))} \\ &\geq C\delta^{-1}|G(x_0,y)| \quad \text{for any } z\in Q_k^\delta \text{ and any } y\in I_k^\delta. \end{aligned} \tag{5.33}$$

Thus, for every $z\in Q_k^\delta$ and $q_0\in(0,\infty)$ we have

$$\left(\fint_{c\,I_k^\delta}|G(x_0,y)|^{q_0}\,dy\right)^{1/q_0} \leq C\delta\, m(z) \tag{5.34}$$

so that, ultimately, for any $q_0,q_1\in(0,\infty)$,

$$\left(\fint_{c\,I_k^\delta}|G(x_0,y)|^{q_0}\,dy\right)^{1/q_0} \leq C\delta\left(\fint_{c\,Q_k^\delta}m(z)^{q_1}\,d\sigma(z)\right)^{1/q_1}, \tag{5.35}$$

after averaging in $z\in Q_k^\delta$. We may then write

$$\begin{aligned} \frac{1}{\delta}\int_{\substack{\frac{\delta}{2}\leq \mathrm{dist}(y,\partial\Omega)\leq\delta \\ y\in\Omega}} &|\nabla_y G(x_0,y)|\,|u(y)|\,dy \leq \sum_{k\in\mathcal{J}_\delta}\frac{1}{\delta}\int_{I_k^\delta}|\nabla_y G(x_0,y)|\,|u(y)|\,dy \\ &\leq \sum_{k\in\mathcal{J}_\delta}\delta^{n-1}\left(\fint_{I_k^\delta}|\nabla_y G(x_0,y)|\,dy\right)\left(\sup_{I_k^\delta}|u|\right) \\ &\leq \sum_{k\in\mathcal{J}_\delta}\delta^{n-1}\left(\fint_{I_k^\delta}|\nabla_y G(x_0,y)|^2\,dy\right)^{1/2}\left(\fint_{c\,I_k^\delta}|u(y)|^p\,dy\right)^{1/p} \\ &\leq \sum_{k\in\mathcal{J}_\delta}\delta^{(n-1)/p'}\left(\fint_{c\,I_k^\delta}\left(\frac{|G(x_0,y)|}{\delta}\right)^2 dy\right)^{1/2}\left(\frac{1}{\delta}\int_{\mathcal{O}_\delta}|(\mathbf{1}_{I_k^\delta}u)(y)|^p\,dy\right)^{1/p} \\ &\leq \sum_{k\in\mathcal{J}_\delta}\left(\int_{c\,Q_k^\delta}m(z)^{p'}\,d\sigma(z)\right)^{1/p'}\left(\int_{c\,Q_k^\delta}|\mathcal{N}_\kappa^\delta u|^p\,d\sigma\right)^{1/p}. \end{aligned} \tag{5.36}$$

Above, the third inequality is a consequence of Hölder's inequality for $\nabla_y G(x_0,y)$ and the L^p-submean inequality for the harmonic function u on I_k^δ, the fourth inequality uses Caccioppoli's estimate for the harmonic function $\nabla[G(x_0,\cdot)]$ on the cube I_k^δ, while the last inequality follows from (5.35) with $q_0=2$ and $q_1=p'$, (2.12) and the fact that there exists $c\geq 1$ such that

$$\operatorname{supp}\mathcal{N}_\kappa^\delta(\mathbf{1}_{I_k^\delta}u)\subset c\,Q_k^\delta. \tag{5.37}$$

We therefore obtain

$$\begin{aligned}\frac{1}{\delta}\int\limits_{\substack{\frac{\delta}{2}\leq\mathrm{dist}(y,\partial\Omega)\leq\delta\\ y\in\Omega}} & |\nabla_y G(x_0,y)|\,|u(y)|\,dy\\ &\leq\sum_{k\in\mathcal{J}_\delta}\left(\int_{c\,Q_k^\delta} m(z)^{p'}\,d\sigma(z)\right)^{1/p'}\left(\int_{c\,Q_k^\delta}|\mathcal{N}_\kappa^\delta u|^p\,d\sigma\right)^{1/p}\\ &\leq C\left(\int_{\partial\Omega} m(z)^{p'}\,d\sigma(z)\right)^{1/p'}\left(\int_{\partial\Omega}|\mathcal{N}_\kappa^\delta u|^p\,d\sigma\right)^{1/p}, \qquad (5.38)\end{aligned}$$

by the discrete version of Hölder's inequality and (5.32). Replacing δ by $2^{-j}\delta$, with $j\geq 0$, in (5.38) then yields

$$\begin{aligned}\frac{1}{\delta}\int\limits_{\substack{2^{-j-1}\delta\leq\mathrm{dist}(y,\partial\Omega)\leq 2^{-j}\delta\\ y\in\Omega}} & |\nabla_y G(x_0,y)|\,|u(y)|\,dy \qquad (5.39)\\ &\leq C2^{-j}\left(\int_{\partial\Omega} m(z)^{p'}\,d\sigma(Z)\right)^{1/p'}\left(\int_{\partial\Omega}|\mathcal{N}_\kappa^\delta u|^p\,d\sigma\right)^{1/p},\end{aligned}$$

so that further, by summing up (5.39) for $j=0,1,\dots,$

$$\frac{1}{\delta}\int\limits_{\mathcal{O}_\delta}|\nabla_y G(x_0,y)|\,|u(y)|\,dy\leq C\left(\int_{\partial\Omega} m(z)^{p'}\,d\sigma(z)\right)^{1/p'}\left(\int_{\partial\Omega}|\mathcal{N}_\kappa^\delta u|^p\,d\sigma\right)^{1/p}. \qquad (5.40)$$

This then entails

$$|I|\leq C\left(\int_{\partial\Omega} m(z)^{p'}\,d\sigma(z)\right)^{1/p'}\left(\int_{\partial\Omega}|\mathcal{N}_\kappa^\delta u|^p\,d\sigma\right)^{1/p}. \qquad (5.41)$$

Switching our attention to II we begin by estimating, while retaining notation introduced above,

$$\begin{aligned}\frac{1}{\delta^2}\int\limits_{\substack{\frac{\delta}{2}\leq\mathrm{dist}(y,\partial\Omega)\leq\delta\\ y\in\Omega}} & |G(x_0,y)|\,|u(y)|\,dy\leq\sum_{k\in\mathcal{J}_\delta}\frac{1}{\delta^2}\int_{I_k^\delta}|G(x_0,y)|\,|u(y)|\,dy\\ &\leq\sum_{k\in\mathcal{J}_\delta}\frac{1}{\delta^2}\left(\int_{I_k^\delta}|G(x_0,y)|^{p'}\,dy\right)^{1/p'}\left(\int_{I_k^\delta}|u(y)|^p\,dy\right)^{1/p}\\ &\leq\sum_{k\in\mathcal{J}_\delta}\left(\int_{c\,Q_k^\delta} m(z)^{p'}\,dZ\right)^{1/p'}\left(\frac{1}{\delta}\int_{\mathcal{O}_\delta}|(\mathbf{1}_{I_k^\delta}u)(y)|^p\,dy\right)^{1/p}\\ &\leq\sum_{k\in\mathcal{J}_\delta}\left(\int_{c\,Q_k^\delta} m(z)^{p'}\,d\sigma(z)\right)^{1/p'}\left(\int_{c\,Q_k^\delta}|\mathcal{N}_\kappa^\delta u|^p\,d\sigma\right)^{1/p}\\ &\leq C\left(\int_{\partial\Omega} m(z)^{p'}\,d\sigma(z)\right)^{1/p'}\left(\int_{\partial\Omega}|\mathcal{N}_\kappa^\delta u|^p\,d\sigma\right)^{1/p}, \qquad (5.42)\end{aligned}$$

by proceeding much as before (note that, this time, (5.35) is used with $q_0=q_1=p'$).

The above estimate is the analogue of (5.38). With this in hand, the same type of argument which has led to (5.41) then yields

$$|II| \leq C\left(\int_{\partial\Omega} m(z)^{p'}\,d\sigma(z)\right)^{1/p'}\left(\int_{\partial\Omega}|\mathcal{N}_\kappa^\delta u|^p\,d\sigma\right)^{1/p}. \tag{5.43}$$

Altogether, (5.24), (5.41) and (5.43) show that

$$|u(x_0)| \leq C(\Omega,\kappa,p,x_0)\,\|\mathcal{N}_\kappa^\delta u\|_{L^p(\partial\Omega,d\sigma)}. \tag{5.44}$$

Since $\lim_{\delta\to 0^+}\mathcal{N}_\kappa^\delta u = 0$ in $L^p(\partial\Omega,d\sigma)$ by the assumptions on u and Lebesgue's Dominated Convergence Theorem, we may finally conclude that $u(x_0)=0$. Given that the point $x_0\in\Omega$ was arbitrary, this shows that the problem (1.4) has a unique solution. The proof of the theorem is therefore finished. □

Having established the above two theorems, it is then a simple matter to present the

Proof of Theorem 1.2. This is an immediate consequence of Theorem 5.1 and Theorem 5.3. □

Next, we record the

Proof of Theorem 1.1. The existence part is established much as in the proof of Theorem 10.1 on pp. 132–133 of [14], which deals with the case of BMO_1 domains. However, for the sake of completeness (and since the proof of uniqueness does not appear to have been explicitly given in [14]), we include Jerison and Kenig's brief argument here. Specifically, if $x_o\in\Omega$ is fixed and $k := \frac{d\omega^{x_o}}{d\sigma}\in B_q(d\sigma)$, some $q\in(1,\infty)$, then $k\in L^q(\partial\Omega,d\sigma)$ and $d\omega^{x_o}=k\,d\sigma$. Set $p_* := q'$, the conjugate exponent of q. Then, if $p>p_*$ and $f\in L^p(\partial\Omega,d\sigma)$, it follows that $f\in L^1(\partial\Omega,d\omega^{x_o})$ by Hölder's inequality. Granted this, Theorem 3.12 then guarantees that if u is as in (3.14) then (3.15)–(3.16) hold. Since, in the current context, the Hardy-Littlewood maximal operator $\mathcal{M}_{\omega^{x_o}}$ is bounded on $L^p(\partial\Omega,d\sigma)$, by a theorem of Muckenhoupt (cf. (vii) in Theorem 6.2), and since ω^{x_o} is mutually absolutely continuous with respect to σ, we may conclude that u solves (1.4) and satisfies (1.5). Finally, showing that a solution to (1.4) with $p>p^*$ is unique can be done by relying on the above argument and Theorem 5.3. □

We conclude this section with the following observation.

Remark 5.4. Retain the context of Theorem 1.2. Comparison of (1.6) and (3.4) yields

$$-\partial_{\nu(x)}G(x,y) = \frac{d\omega^y}{d\sigma}(x), \qquad \forall\, y\in\Omega,\ \ \sigma\text{-a.e. } x\in\partial\Omega. \tag{5.45}$$

Hence, since for any $y\in\Omega$ the tangential gradient of $G(\cdot,y)$ vanishes on the boundary, we may conclude that

$$\lim_{\substack{x\to z\\ x\in\Gamma_\alpha(z)}} \nabla_x G(x,y) = -(d\omega^y/d\sigma)(z)\,\nu(z), \qquad \forall\, y\in\Omega,\ \ \sigma\text{-a.e. } z\in\partial\Omega. \tag{5.46}$$

6. Appendix

In this appendix we record some background results, pertaining to the theory of weights, which are relevant for this paper. Consider a space of homogeneous type (Σ, d, μ). In particular, μ is a measure on Σ satisfying the doubling condition

$$\mu(\Delta_{2r}(x)) \leq C\mu(\Delta_r(x)), \tag{6.1}$$

where $\Delta_r(x) := \{y \in \Sigma : d(x,y) < r\}$ is the ball of radius $r > 0$ centered at $x \in \Sigma$. Some of the properties of the Muckenhoupt class A_∞ in this setting are summarized below.

Definition 6.1. A nonnegative measure $\tilde{\mu}$ on Σ is in $A_\infty(d\mu)$ if, given $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that if $E \subset \Delta_r(x)$, for some arbitrary ball $\Delta_r(x)$, then

$$\frac{\mu(E)}{\mu(\Delta_r(y))} < \delta \Longrightarrow \frac{\tilde{\mu}(E)}{\tilde{\mu}(\Delta_r(y))} < \varepsilon. \tag{6.2}$$

The main properties of the class of weights $A_\infty(d\mu)$ are summarized in the following theorem (see [2], [22] and [15, Theorem 1.4.13, pp. 17–18]).

Theorem 6.2. *Let (Σ, d, μ) be a space of homogeneous type, and assume that $\tilde{\mu}$ is a nonnegative measure on Σ. Then the following are true.*

(i) *If $\tilde{\mu} \in A_\infty(d\mu)$, then $\tilde{\mu}$ is absolutely continuous with respect to μ.*

(ii) *$\tilde{\mu} \in A_\infty(d\mu)$ if and only if $\mu \in A_\infty(d\tilde{\mu})$. Thus, if $\tilde{\mu} \in A_\infty(d\mu)$ then $d\mu$ and $d\tilde{\mu}$ are mutually absolute continuous.*

(iii) *$\tilde{\mu} \in A_\infty(d\mu)$ if and only if there exist $\varepsilon \in (0,1)$, $\delta \in (0,1)$ such that*

$$\frac{\mu(E)}{\mu(\Delta_r(y))} < \delta \Longrightarrow \frac{\tilde{\mu}(E)}{\tilde{\mu}(\Delta_r(y))} < \varepsilon, \quad \forall\, E \subset \Delta_r(y). \tag{6.3}$$

(iv) *$\tilde{\mu} \in A_\infty(d\mu)$ if and only if there exist $C > 0$, $\eta > 0$, $\theta > 0$, such that for every $E \subset \Delta_r(y)$ we have*

$$\frac{\tilde{\mu}(E)}{\tilde{\mu}(\Delta_r(y))} \leq C\Big(\frac{\mu(E)}{\mu(\Delta_r(y))}\Big)^\theta \quad \text{and} \quad \frac{\mu(E)}{\mu(\Delta_r(y))} \leq C\Big(\frac{\tilde{\mu}(E)}{\tilde{\mu}(\Delta_r(y))}\Big)^\eta \tag{6.4}$$

(v) *$A_\infty(d\mu) = \bigcup_{q>1} B_q(d\mu)$, where $\tilde{\mu} \in B_q(d\mu)$ if $\tilde{\mu}$ is absolutely continuous with respect to μ, and the Radon-Nikodym derivative $k := \frac{d\tilde{\mu}}{d\mu}$ belongs to $L^q(\Sigma, d\mu)$, and verifies the reverse Hölder condition*

$$\Big(\frac{1}{\mu(\Delta_r(y))}\int_{\Delta_r(y)} k^q \, d\mu\Big)^{\frac{1}{q}} \leq C\frac{1}{\mu(\Delta_r(y))}\int_{\Delta_r(y)} k \, d\mu, \tag{6.5}$$

for all surface balls $\Delta_r(y)$.

(vi) *If* $\tilde{\mu} \in B_q(d\mu)$ *for some* $q > 1$*, then there exists* $\varepsilon > 0$ *such that* $\tilde{\mu} \in B_{q+\varepsilon}(d\mu)$.

(vii) $\tilde{\mu} \in B_q(d\mu)$ *if and only if the Hardy-Littlewood maximal operator*

$$\mathcal{M}_{\tilde{\mu}}(f)(y) := \sup_{\Delta \ni y,\ \Delta \text{ ball}} \Big(\frac{1}{\tilde{\mu}(\Delta)} \int_{\Delta} |f| \, d\tilde{\mu} \Big), \tag{6.6}$$

satisfies

$$\|\mathcal{M}_{\tilde{\mu}} f\|_{L^{q'}(\Sigma, d\mu)} \leq C \|f\|_{L^{q'}(\Sigma, d\mu)}, \qquad \tfrac{1}{q} + \tfrac{1}{q'} = 1. \tag{6.7}$$

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On Negative Spectrum of Schrödinger Type Operators

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To our dear friend Vladimir Maz'ya

Abstract. The classical Cwikel-Lieb-Rozenblum and Lieb-Thirring inequalities in their general form allow one to estimate the number of negative eigenvalues $N = \#\{\lambda_i < 0\}$ and the sums $S_\gamma = \Sigma|\lambda_i|^\gamma$ for a wide class of Schrödinger operators. We will present here some new examples (Anderson Hamiltonian, operators on lattices, quantum graphs and groups). In some cases below, the parabolic semigroup has an exponential fractional decay at $t \to \infty$. This makes it possible to consider potentials decaying very slowly (logarithmical) at infinity. We also will discuss the case of small local dimension of the underlying manifold, which is usually not covered by the general theory.

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1. Introduction

Let us recall the classical estimates for the number of negative eigenvalues of the operator $H = -\Delta + V(x)$ on $L^2(R^d)$, $d \geq 3$. Let $N_E(V)$ be the number of eigenvalues E_i of the operator H that are below or equal to $E \leq 0$. In particular, $N_0(V)$ is the number of non-positive eigenvalues. Let $N(V) = \#\{E_i < 0\}$ be the number of strictly negative eigenvalues of the operator H. Then the Cwikel-Lieb-Rozenblum and Lieb-Thirring inequalities have the following form, respectively, (see [4], [9]–[12], [15])

$$N(V) \leq C_d \int_{R^d} W^{\frac{d}{2}}(x)dx, \tag{1.1}$$

$$\sum_{i:E_i<0} |E_i|^\gamma \leq C_{d,\gamma} \int_{R^d} W^{\frac{d}{2}+\gamma}(x)dx. \tag{1.2}$$

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Here $W = V_- = \max(0, -V(x))$, $d \geq 3$, $\gamma \geq 0$. Inequality (1.1) can be considered as a particular case of (1.2) with $\gamma = 0$. Conversely, inequality (1.2) can be easily derived from (1.1) (see [14]). So, below we will mostly discuss the Cwikel-Lieb-Rozenblum inequality and its extensions.

A review of different approaches to the proof of (1.1) can be found in [17]. The most general form of this inequality has the following form. Let (X, μ) be a measure space with a σ-finite measure $\mu = \mu(dx)$. Let H_0 be a non-negative self-adjoint operator in $L^2(X, \mu)$ such that for each $t > 0$ the operator e^{-tH_0} is bounded and positivity preserving. Let $p_0(t, x, y)$ be the kernel of e^{-tH_0}. There is a natural way to define the values $p_0(t, x, x)$ of the kernel on the diagonal for almost all $x \in X$. Suppose that there is a function $\pi(t)$ such that for almost all $x \in X$,

$$p_0(t, x, x) \leq \pi(t). \tag{1.3}$$

Then

$$N_0(V) \leq \frac{1}{g(1)} \int_0^\infty \frac{\pi(t)}{t} dt \int_X G(tW(x))\mu(dx), \tag{1.4}$$

$$\sum_{i:E_i<0} |E_i|^\gamma \leq \frac{1}{g(1)} \int_0^\infty \frac{\pi(t)}{t} dt \int_X G(tW(x))W(x)^\gamma \mu(dx) \tag{1.5}$$

for any continuous, convex, non-negative function G which grows not faster than a polynomial at infinity, and is such that $z^{-1}G(z)$ is integrable at zero (hence, $G(0) = 0$), and the integral (1.4) is finite. The function $g(\lambda)$, $\lambda \geq 0$, is defined by

$$g(\lambda) = \int_0^\infty z^{-1}G(z)e^{-z\lambda}dz, \quad \text{i.e., } g(1) = \int_0^\infty z^{-1}G(z)e^{-z}dz. \tag{1.6}$$

Note that $\pi(t) = (2\pi t)^{-\frac{d}{2}}$ in the classical case of $H_0 = -\Delta$ on $L^2(R^d)$, and (1.1) follows from (1.4) in this case by substitution $t \to \tau = tW(x)$ if G is such that $\int_0^\infty z^{-1-\frac{d}{2}}G(z)dz < \infty$.

The classical Lieb approach [9]–[12] (based on the Kac-Feinman formula) was extended by I. Daubichies [5] to prove (1.4), (1.5) for some pseudo-differential operators in R^d. It is also mentioned there that the Lieb method works in a wider setting. A slightly different approach based on the Trotter formula was used by G. Rozenblum and M. Solomyak [16] (see also [17] and [5]) to prove (1.4) when X is a measure space. All these papers contain estimate (1.4) for $N(V)$, not for $N_0(V)$. Its validity for $N_0(V)$ is shown in [13]. The latter paper contains also a derivation of (1.4), (1.5) for operators on functions in metric spaces X using the Lieb method, as well as many examples. Note that verifying assumption (1.3) with exact bounds for $\pi(t)$ can be a challenging problem in some cases.

The inequalities above are meaningful only for those W for which integrals converge. They become particularly transparent if $G(z) = 0$ for $z \leq \sigma$, $G(z) = z-\sigma$

for $z > \sigma$, $\sigma \geq 0$. Then (1.4), (1.5) take the form

$$N_0(V) \leq \frac{1}{c(\sigma)} \int_X W(x) \int_{\frac{\sigma}{W(x)}}^{\infty} \pi(t) dt \mu(dx), \tag{1.7}$$

$$\sum_{i:E_i<0} |E_i|^\gamma \leq \frac{1}{c(\sigma)} \int_X W^{\gamma+1}(x) \int_{\frac{\sigma}{W(x)}}^{\infty} dt \mu(dx), \tag{1.8}$$

where $c(\sigma) = e^{-\sigma} \int_0^\infty \frac{ze^{-z}dz}{z+\sigma}$.

Inequalities (1.4), (1.5) or (1.7), (1.8) allow one to study negative eigenvalues of operators for which the function $\pi(t)$ has different power asymptotics as $t \to 0$ and $t \to \infty$. Let

$$p_0(t, x, x) \leq c/t^{\alpha/2}, \ t \leq h, \ p_0(t, x, x) \leq c/t^{\beta/2}, \ t > h, \tag{1.9}$$

where $h > 0$ is arbitrary. The parameters α and β characterize the "local dimension" and the "global dimension" of X, respectively. For example, $\alpha = \beta = d$ in the classical case of the Laplacian $H_0 = -\Delta$ in the Euclidean space $X = R^d$. If $H_0 = -\Delta$ is the difference Laplacian on the lattice $X = Z^d$, then $\alpha = 0$, $\beta = d$. If $X = S^n \times R^d$ is the product of n-dimensional sphere and R^d, then $\alpha = n + d$, $\beta = d$.

If $\alpha, \beta > 2$, inequality (1.4) implies (see [17]) that

$$N_0(V) \leq C(h) \left[\int_{\{W(x) \leq h^{-1}\}} W^{\frac{\beta}{2}}(x) \mu(dx) + \int_{\{W(x) > h^{-1}\}} W^{\frac{\alpha}{2}}(x) \mu(dx) \right]. \tag{1.10}$$

Note that the restriction $\beta > 2$ is essential here in the same way as the condition $d > 2$ in (1.1). We will show that the assumption on α can be omitted, but the form of the estimate in (1.10) changes in this case.

The main goal of the paper is a new set of examples. We will consider operators which may have different power asymptotics of $\pi(t)$ as $t \to 0$ or $t \to \infty$ or exponential asymptotics as $t \to \infty$. The latter case will allow us to consider the potentials which decay very slowly at infinity. This is particularly important in some applications, such as Anderson model, where the borderline between operators with a finite and infinite number of eigenvalues is defined by the decay of the perturbation in a logarithmic scale.

We have the following two consequences of (1.7)

Theorem 1.1. *Let H_0 be a non-negative self-adjoint operator in $L^2(X, \mu)$, the operator e^{-tH_0}, $t > 0$, be bounded and positivity preserving, and* (1.3) *hold. Let*

$$\pi(t) \leq c/t^{\beta/2}, \quad t \to \infty; \quad \pi(t) \leq c/t^{\alpha/2}, \quad t \to 0 \tag{1.11}$$

for some $\beta > 2$ and $\alpha \geq 0$. Then

$$N_0(V) \leq C(h) \left[\int_{X_h^-} W(x)^{\beta/2} \mu(dx) + \int_{X_h^+} bW(x)^{\max(\alpha/2,1)} \mu(dx) \right], \tag{1.12}$$

where $X_h^- = \{x : W(x) \leq h^{-1}\}$, $X_h^+ = \{x : W(x) > h^{-1}\}$, $b = 1$ if $\alpha \neq 2$, $b = ln(1 + W(x))$ if $\alpha = 2$.

Remark. In some cases $\max(\alpha/2, 1)$ can be replaced by $\alpha/2$, as will be discussed in Section 2.

Proof of Theorem 1.1. We write (1.7) in the form $N_0(V) \le I_- + I_+$, where $I_\mp$ correspond to integration in (1.7) over $X_h^\mp$, respectively.

Let $x \in X_h^-$, i.e., $W < h^{-1}$. Then the interior integral in (1.7) does not exceed

$$C(h)\int_{\frac{\sigma}{W}}^{\infty} t^{-\beta/2}dt = C(h)W^{(\beta/2)-1}. \tag{1.13}$$

Thus I_- can be estimated by the first term in the right-hand side of (1.12). Similarly, I_+ can be estimated from above by

$$C(h)\int_{X_h^+} W\Big(\int_{\frac{\sigma}{W}}^{h} + \int_h^\infty\Big)\pi(t)dt \le C(h)\int_{X_h^+} W\Big(\int_{\frac{\sigma}{W}}^{h} t^{-\alpha/2}dt + \int_h^\infty t^{-\beta/2}dt\Big)dx,$$

which does not exceed the second term in the right-hand side of (1.12). □

Theorem 1.2. *Let H_0 be a non-negative self-adjoint operator in $L^2(X,\mu)$, the operator e^{-tH_0}, $t>0$, be bounded and positivity preserving, and* (1.3) *hold. Let*

$$\pi(t) \le ce^{-at^\gamma}, \quad t\to\infty; \qquad \pi(t) \le c/t^{\alpha/2}, \quad t \to 0 \tag{1.14}$$

for some $\gamma > 0$ and $\alpha \ge 0$. Then for each $A > 0$,

$$N_0(V) \le C(h,A)\Big[\int_{X_h^-} e^{-AW(x)^{-\gamma}}\mu(dx) + \int_{X_h^+} bW(x)^{\max(\alpha/2,1)}\mu(dx)\Big], \tag{1.15}$$

where X_h^-, X_h^+, b are the same as in the theorem above.

The proof is the same as that of the theorem above. One only needs to replace (1.13) by the following estimate

$$C(h)\int_{\frac{\sigma}{W}}^\infty e^{-at^\gamma}dt = C(h)W^{-1}\int_\sigma^\infty e^{-a(\frac{\tau}{W})^\gamma}d\tau \le C(h)W^{-1}e^{-\frac{a}{2}(\frac{\sigma}{W})^\gamma}\int_\sigma^\infty e^{-\frac{a}{2}(\frac{\tau}{W})^\gamma}d\tau$$
$$\le \Big[C(h)W^{-1}\int_\sigma^\infty e^{-\frac{a}{2}(h\tau)^\gamma}d\tau\Big]e^{-\frac{a}{2}(\frac{\sigma}{W})^\gamma},$$

and note that σ can be chosen as large as we please.

The next sections contain some applications of the results discussed above.

2. Small local dimension ($\alpha = 0, 1$)

2.1 Operators on lattices and discrete groups. It is easy to see that Theorems 1.1 and 1.2 are not exact if $\alpha \le 2$. We are going to illustrate this fact now and provide a better result for the case $\alpha = 0$ which occurs, for example, when operators on lattices and discrete groups are considered. An important example with $\alpha = 1$ will be discussed in the next section (operators on quantum graphs).

Let $X = \{x\}$ be a countable set and H_0 be a difference operator on $L^2(X)$ which is defined by

$$(H_0\psi)(x) = \sum_{y\in X} a(x,y)\psi(y), \tag{2.1}$$

where

$$a(x,x) > 0,\ a(x,y) = a(y,x) \le 0,\ \sum_{y\in X} a(x,y) = 0.$$

A typical example of H_0 is the negative difference Laplacian on the lattice $X = Z^d$, i.e.,

$$(H_0\psi)(x) = -\Delta\psi = \sum_{y\in Z^d:|y-x|=1} [\psi(x) - \psi(y],\ x \in Z^d. \tag{2.2}$$

We will assume that $0 < a(x,x) \le c_0 < \infty$. Then $\mathrm{Sp}H_0 \subset [0, 2c_0]$. The operator $-H_0$ defines the Markov chain $x(s)$ on X with continuous time $s \ge 0$ which spends exponential time with parameter $a(x,x)$ at each point $x \in X$ and then jumps to a point $y \in X$ with probability $r(x,y) = \frac{a(x,y)}{a(x,x)}$, $\sum_{y:y\ne x} r(x,y) = 1$. The transition matrix $p(t,x,y) = P_x(x_t = y)$ is the fundamental solution of the parabolic problem

$$\frac{\partial p}{\partial t} + H_0 p = 0,\ p(0,x,y) = \delta_y(x).$$

Obviously, $p(t,x,x) \le \pi(t) \le 1$, and $\pi(t) \to 1$ uniformly in x as $t \to 0$. The asymptotic behavior of $\pi(t)$ as $t \to \infty$ depends on operator H_0 and can be more or less arbitrary.

Consider now the operator $H = H_0 - m\delta_y(x)$ with the potential supported on one point. The negative spectrum of H contains at most one eigenvalue (due to rank one perturbation arguments), and such an eigenvalue exists if $m \ge c_0$. The latter follows from the variational principle, since

$$\langle H_0\delta_y,\ \delta_y\rangle - m\,\langle\delta_y,\ \delta_y\rangle \ \le\ c_0 - m\ < 0.$$

However, Theorems 1.1 and 1.2 estimate the number of negative eigenvalues $N(V)$ of the operator H by Cm. Similarly, if

$$V = -\sum_{1\le i\le n} m_i\delta(x - x_i)$$

and $m_i \ge c_0$, then $N(V) = n$, but Theorems 1.1 and 1.2 give only that $N(V) \le C\sum m_i$. The following statement provides a better result for the case under consideration. The meaning of the statement below is that we replace $\max(\alpha/2, 1) = 1$ in (1.12), (1.15) by $\alpha/2 = 0$. Let us also mention that these theorems can not be strengthened in a similar way if $0 < \alpha \le 2$ (see Section 2.1).

Theorem 2.1. *Let $H = H_0 + V(x)$, where H_0 is defined in* (2.1). *Then for each $h > 0$,*

$$N_0(V) \le C(h)[n(h) + \int_0^\infty \frac{\pi(t)}{t} \sum_{x\in X_h^-} G(tW(x))dt],\ n(h) = \#\{x \in X_h^+\}.$$

If, additionally, either (1.11) *or* (1.14) *is valid for* $\pi(t)$ *as* $t \to \infty$, *then for each* $A > 0$,

$$N_0(V) \le C(h)\Big[\sum_{x\in X_h^-} W(x)^{\frac{\beta}{2}} + n(h)\Big],\ n(h) = \#\{x \in X_h^+\},$$

$$N_0(V) \le C(h,A)\Big[\sum_{x\in X_h^-} e^{-AW(x)^{-\gamma}} + n(h)\Big],\ n(h) = \#\{x \in X_h^+\},$$

respectively.

Remark. This theorem is a simple generalization of a similar statement for operator (2.2) proved in [16].

Proof. In order to prove the first inequality, we split the potential $V(x) = V_1(x) + V_2(x)$, where $V_2(x) = V(x)$ for $x \in X_h^+$, $V_2(x) = 0$ for $x \in X_h^-$. Now for each $\varepsilon \in (0,1)$,

$$N_0(V) \le N_0(\varepsilon^{-1}V_1) + N_0((1-\varepsilon)^{-1}V_2) = N_0(\varepsilon^{-1}V_1) + n(h). \tag{2.3}$$

It remains to apply (1.4) to the operator $-\Delta + \varepsilon^{-1}V_1$ and pass to the limit as $\varepsilon \to 1$. The next two inequalities follow from Theorems 1.1 and 1.2. □

2.2 Operators on quantum graphs. We will consider a specific quantum graph Γ^d, the so-called Avron-Exner-Last graph. Its vertices are the points of the lattice Z^d, and the edges are all segments of length one connecting neighboring vertices. Let $s \in [0,1]$ be the natural parameter on the edges (distance from one of the end points of the edge). Consider the space D of smooth functions φ on edges of Γ^d with the following (Kirchoff's) boundary conditions at vertices: at each vertex φ is continuous and

$$\sum_{i=1}^{d} \varphi_i' = 0, \tag{2.4}$$

where φ_i' are the derivatives along the adjoint edges in the direction out of the vertex. The operator H_0 acts on functions $\varphi \in D$ as $-\frac{d^2}{ds^2}$. The closure of this operator in $L^2(\Gamma^d)$ is a self-adjoint operator with the spectrum $[0,\infty)$ (see [3]).

Theorem 2.2. *The assumptions of Theorem* 1.1 *hold for operator* $H_0 = -\frac{d^2}{ds^2}$ *on the graph* Γ^d *with the constants* α, β *in Theorem* 1.1 *equal to* 1 *and* d, *respectively.*

Proof. One can easily see that there is a Markov process with the generator $-H_0$. Thus one needs only to estimate the function p_0 and find constants α, β.

Let

$$u_t = -H_0 u,\ t > 0,\ u|_{t=0} = f,$$

with a compactly supported f and

$$\varphi = \varphi(x,\lambda) = \int_0^\infty u e^{\lambda t} dt,\ \mathrm{Re}\lambda \le -a < 0,\ x \in \Gamma^d.$$

Note that we replaced $-\lambda$ by λ in the Laplace transform above. It is convenient for future notations. Then φ satisfies the equation

$$(H_0 - \lambda)\varphi = f, \tag{2.5}$$

and u can be found using the inverse Laplace transform

$$u = \frac{1}{(2\pi)^d} \int_{-a-i\infty}^{-a+i\infty} \varphi e^{-\lambda t} d\lambda. \tag{2.6}$$

The spectrum of H_0 is $[0, \infty)$, and φ is analytic in λ when $\lambda \in C \backslash [0, \infty)$. We are going to study the properties of φ when $\lambda \to 0$ and $\lambda \to \infty$. Let $\psi(z) = \psi(z, \lambda)$, $z \in Z^d$, be the restriction of the function $\varphi(x, \lambda),\ x \in \Gamma^d$, on the lattice Z^d. Let e be an arbitrary edge of Γ^d with end points $z_1, z_2 \in Z^d$ and parametrization from z_1 to z_2. By solving the boundary value problem on e, we can represent φ on e in the form

$$\varphi = \frac{\psi(z_1)\sin k(1-s) + \psi(z_2)\sin ks}{\sin k} + \varphi_{\text{par}},\ \varphi_{\text{par}} = \int_0^1 G(s,t) f(t) dt, \tag{2.7}$$

where $k = \sqrt{\lambda}$, $\text{Im}k > 0$, and

$$G = \frac{1}{k \sin k} \begin{cases} \sin ks \sin k(1-t), & s < t \\ \sin kt \sin k(1-s), & s \geq t. \end{cases}$$

Due to the invariance of H_0 with respect to translations and rotations in Z^d, it is enough to estimate $p_0(t, x, x)$ when x belongs to the edge e_0 with z_1 being the origin in Z^d and $z_2 = (1, 0, \ldots, 0)$. Let f be supported on one edge e_0. Then (2.7) is still valid, but $\varphi_{par} = 0$ on all the edges except e_0. We substitute (2.7) into (2.4) and get the following equation for ψ:

$$(\Delta - 2d \cos k)\psi(z) = \frac{1}{k} \int_0^1 \sin k(1-t) f(t) dt \delta_1 + \frac{1}{k} \int_0^1 \sin kt f(t) dt \delta_0,\ z \in Z^d.$$

Here Δ is the lattice Laplacian defined in (2.2) and δ_0, δ_1 are functions on Z^d equal to one at z, y, respectively, and equal to zero elsewhere. In particular, if f is the delta function at a point s of the edge e_0, then

$$(\Delta - 2d \cos k)\psi = \frac{1}{k} \sin k(1-s)\delta_1 + \frac{1}{k} \sin ks \delta_0. \tag{2.8}$$

Let $R_\mu(z - z_0)$ be the kernel of the resolvent $(\Delta - \mu)^{-1}$ of the lattice Laplacian. Then (2.8) implies that

$$\psi(z) = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} s R_\mu(z) + \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(1-s) R_\mu(z - z_2),\ \mu = 2d \cos \sqrt{\lambda}. \tag{2.9}$$

Function $R_\mu(z)$ has the form

$$R_\mu(z) = \int_T \frac{e^{i(\sigma,z)} d\sigma}{(\sum_{1 \leq j \leq d} 2 \cos \sigma_j) - \mu},\ T = [-\pi, \pi]^d.$$

Hence, function $\sin(\sqrt{\lambda}s)R_\mu(z)$, $s \in (0,1)$, $\mu = 2d\cos\sqrt{\lambda}$, decays exponentially as $|\mathrm{Im}\sqrt{\lambda}| \to \infty$. This allows one to change the contour of integration in (2.6), when $z \in Z^d$, and rewrite (2.6) in the form

$$u(z,t) = \frac{1}{(2\pi)^d}\int_l \psi_\lambda(z)e^{-\lambda t}d\lambda,\ z \in Z^d, \tag{2.10}$$

where contour l consists of the ray $\lambda = \rho e^{-i\pi/4},\ \rho \in (\infty,1)$, a smooth arc starting at $\lambda = e^{-\pi/4}$, ending at $\lambda = e^{\pi/4}$, and crossing the real axis at $\lambda = -a$, and the ray $\lambda = \rho e^{i\pi/4},\ \rho \in (1,\infty)$. It is easy to see that $|\psi(z,\lambda)| \leq C/|\sqrt{\lambda}|$ as $\lambda \in l$ uniformly in s and $z \in Z^d$. This immediately implies that $|u(z,t)| \leq C/\sqrt{t}$. Thus, (1.3) and the second of estimates (1.11) hold with $\alpha = 1$.

From (2.10) it also follows that the asymptotic behavior of u as $t \to \infty$ is determined by the asymptotic expansion of $\psi(z,\lambda)$ as $\lambda \to 0$, $\lambda \notin [0,\infty)$. Note that the spectrum of the difference Laplacian is $[-2d, 2d]$, and $\mu = 2d - d\lambda + O(\lambda^2)$ as $\lambda \to 0$. From here and the well-known expansions of the resolvent of the difference Laplacian near the edge of the spectrum it follows that the first singular term in the asymptotic expansion of $R_\mu(z)$ as $\lambda \to 0$, $\lambda \notin [0,\infty)$, has the form

$$\begin{cases} c_d\lambda^{d/2-1}(1+O(\lambda)), & d \text{ is odd,}\\ c_d\lambda^{d/2-1}\ln\lambda(1+O(\lambda)), & d \text{ is even.}\end{cases}$$

Then (2.9) implies that a similar expansion is valid for $\psi(z,\lambda)$ with the main term independent of s and the remainder estimated uniformly in s. This allows one to replace l in (2.10) by the contour which consists of the rays $\arg\lambda = \pm\pi/4$. From here it follows that for each $z \in Z^d$ and uniformly in s,

$$u(z,t) \sim t^{-d/2},\ t \to \infty.$$

Hence, $\beta = d$. □

As we discussed in the beginning of this section, Theorem 1.1 is not exact if $\alpha \leq 2$. Theorem 2.1 provides a better result in the case $\alpha = 0$. The situation is more complicated if $\alpha = 1$. We will illustrate it using the operator H_0 on quantum graph Γ^d perturbed by a potential. Two specific classes of potentials will be considered. In one case, inequality (1.12) is valid with $\max(\alpha/2, 1) = 1$ replaced by $\alpha/2 = 1/2$. However, inequality (1.12) can not be improved for potentials of the second type. The first class consists of potentials which are constant on each edge.

Theorem 2.3. *Let $d \geq 3$ and V be constant on each edge e_i of the graph: $V(x) = -v_i < 0,\ x \in e_i$. Then*

$$N_0(V) \leq c(h)\Big(\sum_{i:\ v_i \leq h^{-1}} v_i^{d/2} + \sum_{i:\ v_i > h^{-1}} \sqrt{v_i}\Big).$$

Proof. Put $V(x) = V_1(x) + V_2(x)$, where $V_1(x) = V(x)$ if $|V(x)| > h^{-1},\ V_1(x) = 0$ if $|V(x)| \leq h^{-1}$. Then (see (2.3))

$$N_0(V) \leq N_0(2V_1) + N_0(2V_2).$$

One can estimate $N(V_1)$ from above (below) by imposing the Neumann (Dirichlet) boundary conditions at all vertices of Γ. This leads to the estimates

$$\sum_{i:\ v_i>h^{-1}} \frac{\sqrt{2v_i}}{\pi} \le N_0(V) \le \sum_{i:\ v_i>h^{-1}} \left(\frac{\sqrt{2v_i}}{\pi}+1\right) \le c(h) \sum_{i:\ v_i>h^{-1}} \sqrt{v_i},$$

which, together with Theorem 1.1 applied to $N_0(2V_2)$, justifies the statement of the theorem. □

The same arguments allow one to get a more general result.

Theorem 2.4. *Let $d \ge 3$. Let Γ^d_- be the set of edges e_i of the graph Γ^d where $W \le h^{-1}$, Γ^d_+ be the complementary set of edges, and*

$$\frac{\sup_{x\in e_i} W(x)}{\min_{x\in e_i} W(x)} \le k_0 = k_0(h),\ \ x \in \Gamma^d_+,$$

where $W = V_-$. Then

$$N_0(V) \le c(h,k_0)\left(\int_{\Gamma^d_-} W(x)^{d/2}dx + \int_{\Gamma^d_+} \sqrt{W(x)}dx\right).$$

Example. The next example shows that there are singular potentials on Γ^d for which $\max(\alpha/2, 1)$ in (1.12) can not be replaced by any value less than one. Consider the potential $V(x) = -A\sum_{i=1}^m \delta(x-x_i)$, where x_i are middle points of some edges, and $A > 4$. One can easily modify the example by considering δ -sequences instead of δ-functions (in order to get a smooth potential.) Then

$$\int_{\Gamma^d} W^\sigma(x)dx = 0$$

for any $\sigma < 1$, while $N(V) \ge m$. In fact, consider the Sturm-Liouville problem on the interval $[-1/2, 1/2]$:

$$-y'' - A\delta(x)y = \lambda y,\ \ y(-1/2) = y(1/2) = 0,\ \ A > 4.$$

It has a unique negative eigenvalue which is the root of the equation

$$\tanh(\sqrt{-\lambda}/2) = 2\sqrt{-\lambda}/A.$$

The corresponding eigenfunction is $y = \sinh[\sqrt{-\lambda}(|x|+1/2)]$. The estimate $N(V) \ge m$ follows by imposing the Dirichlet boundary conditions on the vertices of Γ^d.

3. Anderson model

3.1 Discrete case. Consider the classical Anderson Hamiltonian $H_0 = -\Delta + V(x,\omega)$ on $L^2(Z^d)$ with random potential $V(x,\omega)$. Here

$$\Delta\psi(x) = \sum_{x':|x'-x|=1} \psi(x') - 2d\psi(x).$$

We assume that random variables $V(x,\omega)$ on the probability space (Ω, F, P) have the Bernoulli structure, i.e., they are i.i.d. and $P\{V(\cdot) = 0\} = p > 0$,

$P\{V(\cdot) = 1\} = q = 1 - p > 0$. The spectrum of H_0 is equal to (see [2])

$$\mathrm{Sp}(H_0) = \mathrm{Sp}(-\Delta) \oplus 1 = [0, 4d + 1].$$

Let us stress that $0 \in \mathrm{Sp}(H_0)$ due to the existence P-a.s. of arbitrarily large clearings in realizations of V, i.e., there are balls $B_n = \{x : |x - x_n| < r_n\}$ such that $V(x) = 0,\ x \in B_n$, and $r_n \to \infty$ as $n \to \infty$ (see the proof of the theorem below for details).

Let

$$H = H_0 - W(x),\ W(x) \geq 0.$$

The operator H has discrete random spectrum on $(-\infty, 0]$ with possible accumulation point at $\lambda = 0$. Put $N_0(-W) = \#\{\lambda_i \leq 0\}$. Obviously, $N_0(-W)$ is random.

Theorem 3.1.

(a) *For each* $h > 0$ *and* $\gamma < \frac{d}{d+2}$,

$$EN_0(-W) \leq c_1(h)[\#\{x \in Z^d : W(x) \geq h^{-1}\}] + c_2(h, \gamma) \sum_{x: W(x) < h^{-1}} e^{-\frac{1}{W^\gamma(x)}}. \tag{3.1}$$

In particular, if $W(x) < \frac{C}{\log^\sigma |x|}$, $|x| \to \infty$, *with some* $\sigma > \frac{d+2}{d}$, *then* $EN_0(-W) < \infty$, *i.e.,* $N_0(-W) < \infty$ *almost surely.*

(b) *If*

$$W(x) > \frac{C}{\log^\sigma |x|},\ |x| \to \infty, \quad \textit{and} \ \ \sigma < \frac{2}{d}, \tag{3.2}$$

then $N_0(-W) = \infty$ *a.s.* (*in particular,* $EN_0(-W) = \infty$).

Proof. Since $W \geq 0$, the kernel $p_0(t, x, y)$ of the semigroup $\exp(t(\Delta + W))$ can be estimated by the kernel of $\exp(t\Delta)$, i.e., by the transition probability of the random walk with continuous time on Z^d. The diagonal part of this kernel $p_0(t, x, x, \omega)$ is a stationary field on Z^d. Due to the Donsker-Varadhan estimate (see [6], [7]),

$$Ep_0(t, x, x, \omega) = Ep_0(t, x, x, \omega) \overset{\log}{\sim} \exp(-c_d t^{\frac{d}{d+2}}), \quad t \to \infty,$$

i.e.,

$$\log Ep_0 \sim -c_d t^{\frac{d}{d+2}}, \quad t \to \infty.$$

On the rigorous level, the relations above must be understood as estimates from above and below, and the upper estimate has the following form: for each $\delta > 0$,

$$Ep_0 \leq C(\delta) \exp(-c_d t^{\frac{d}{d+2} - \delta}), \quad t \to \infty. \tag{3.3}$$

Now the first part of the theorem is a consequence of (1.4) and Theorems 1.2.

The proof of the second part is based on the following lemma which indicates the existence of large clearings at the distances which are not too large. We denote by $C(r)$ the cube in the lattice,

$$C(r) = \{x \in Z^d : |x_i| < r,\ 1 \leq i \leq d\}.$$

Let's divide Z^d into cubic layers $L_n = C(a^{n+1})\backslash C(a^n)$ with some constant $a \geq 1$ which will be selected later. One can choose a set $\Gamma^{(n)} = \{z_i^{(n)} \in L_n\}$ in each layer L_n such that

$$|z_i^{(n)} - z_j^{(n)}| \geq 2n^{\frac{1}{d}} + 1, \qquad d(z_i^{(n)}, \partial L_n) > n^{\frac{1}{d}},$$

and

$$|\Gamma^{(n)}| \geq c\frac{(2a)^{n(d-1)}a^{n+1}}{(2n^{1/d})^d} \geq ca^{nd}, \quad n \to \infty.$$

Let $C(n^{1/d}, i)$ be the cube $C(n^{1/d})$ with the center shifted to the point $z_i^{(n)}$. Obviously, cubes $C_{n^{1/d},i}$ do not intersect each other, $C(n^{1/d}, i) \subset L_n$ and $|C(n^{1/d}, i)| \leq c'n$.

Consider the following event A_n ={each cube $C(n^{1/d}, i) \subset L_n$ contains at least one point where $V(x) = 1$}. Obviously,

$$P(A_n) = (1 - p^{|C(n^{1/d},i)|})^{|\Gamma^{(n)}|} \leq e^{-|\Gamma^{(n)}|p^{|C(n^{1/d},i)|}} \leq e^{-ca^{nd}c'p^n} = e^{-c(a^d p^{c'})^n}.$$

We will choose a big enough, so that $a^d p^{c'} > 1$. Then $\sum P(A_n) < \infty$, and the Borel-Cantelli lemma implies that P-a.s. there exists $n_0(\omega)$ such that each layer L_n, $n \geq n_0(\omega)$, contains at least one empty cube $C(n^{1/d}, i)$, $i = i(n)$. Then from (3.2) it follows that

$$W(x) \geq \frac{C}{n^{\frac{2}{d}-\delta}} = \varepsilon_n, \qquad x \in C(n^{1/d}, i), \qquad i = i(n).$$

One can easily show that the operator $H = -\Delta - \varepsilon$ in a cube $C \subset Z^d$ with the Dirichlet boundary condition at ∂C has at least one negative eigenvalue if $|C|\varepsilon^{d/2}$ is big enough. Thus the operator H in $C(n^{1/d}, i(n))$ with the Dirichlet boundary condition has at least one eigenvalue if n is big enough, and therefore $N(-W) = \infty$. □

Remark. We expect to get a better result than (3.1) using a modification of Sznitman's estimates for the principal eigenvalue of the Dirichlet problem for the Anderson operator in a cube of size $L \to \infty$. Inequality (3.1) will be valid for $N_0(-W)$ P-a.s. (not for $EN_0(-W)$), and with $\gamma < 2/d$, i.e., the decay $W(x) \sim \log^{2/d}|x|$, $|x| \to \infty$, is a borderline between a finite and infinite number of eigenvalues. This result will be published elsewhere.

3.2 Continuous case. Theorem 3.1 is also valid for Anderson operators in R^d. Let $H_0 = -\Delta + V(x,\omega)$ on $L^2(R^d)$ with the random potential

$$V(x,\omega) = \sum_{n \in Z^d} \varepsilon_n I_{Q_n}(x),\ x \in R^d,\ n = (n_1, \ldots, n_d),$$

where $Q_n = \{x \in R^d : n_i \leq x_i < n_i + 1,\ i = 1, 2, \ldots, d\}$ and ε_n are independent Bernoulli r.v. with $P\{\varepsilon_n = 0\} = p$, $P\{\varepsilon_n = 1\} = q = 1-p$. Put $H = H_0 - W(x) = -\Delta + V(x,\omega) - W(x)$.

Theorem 3.2.

(a) *If* $d \geq 3$*, then for each* $h > 0$ *and* $\gamma < \frac{d}{d+2}$,

$$EN_0(-W) \leq c_1(h) \int_{W(x) \geq h^{-1}} W(x)^{d/2} dx + c_2(h, \gamma) \int_{W(x) < h^{-1}} e^{-\frac{1}{W^{\gamma}(x)}} dx.$$

In particular, if $W(x) < \frac{C}{\log^{\sigma}|x|}$, $|x| \to \infty$*, with some* $\sigma > \frac{d+2}{d}$*, then* $EN_0(-W) < \infty$*, i.e.,* $N_0(-W) < \infty$ *almost surely.*

(b) *If*

$$W(x) > \frac{C}{\log^{\sigma}|x|}, \ |x| \to \infty, \quad \text{and } \ \sigma < \frac{2}{d},$$

then $N_0(-W) = \infty$ *a.s. (in particular,* $EN_0(-W) = \infty$*).*

The proof of this theorem is identical to the proof of Theorem 3.1 with the only difference that now $p_0(t, 0, 0)$ is not bounded as $t \to 0$, but $p_0(t, 0, 0) \leq c/t^{d/2}$, $t \to 0$.

4. Continuous and discrete groups

Some applications of Theorems 1.1, 1.2 will be given concerning left invariant operators $H_0 = -\Delta_\Gamma$ on the groups $X = \Gamma$. A Markov process with generator Δ_Γ can be defined in a natural way in examples below, and therefore the operator $e^{t\Delta_\Gamma}$ is positivity preserving. Estimate (1.3) obviously holds, since operator Δ_Γ is invariant and $p(t, x, x)$ does not depend on x. Thus it remains only to estimate $\pi(t)$ at zero and infinity. We will do it in detail in the first example (free groups) and we only state the results in other cases.

4.1 Free groups. Let X be a group Γ with generators $a_1, a_2, \ldots, a_d$, inverse elements $a_{-1}, a_{-2}, \ldots, a_{-d}$, the unit element e, and with no relations between generators except $a_i a_{-i} = a_{-i} a_i = e$. The elements $g \in \Gamma$ are the shortest versions of the words $g = a_{i_1} \cdot \cdots \cdot a_{i_n}$ (with all factors e and $a_j a_{-j}$ being omitted). The metric on Γ is given by

$$d(g_1, g_2) = d(e, g_1^{-1} g_2) = m(g_1^{-1} g_2),$$

where $m(g)$ is the number of letters $a_{\pm i}$ in g. The measure μ on Γ is defined by $\mu(\{g\}) = 1$ for each $g \in \Gamma$. It is easy to see that $|\{g : d(e, g) = R\}| = 2d(2d-1)^{R-1}$, i.e., the group Γ has an exponential growth rate.

Define the operator Δ_Γ on $X = \Gamma$ by the formula

$$\Delta_\Gamma \psi(g) = \sum_{-d \leq i \leq d, \ i \neq 0} [\psi(g a_i) - \psi(g)]. \tag{4.1}$$

Obviously, the operator $-\Delta_\Gamma$ is bounded and non-negative in $L^2(\Gamma, \mu)$. In fact, $||\Delta_\Gamma|| \leq 4d$. As it is easy to see, the operator Δ_Γ is left-invariant:

$$(\Delta_\Gamma \psi)(gx) = \Delta_\Gamma(\psi(gx)), \ x \in \Gamma,$$

for each fixed $g \in \Gamma$. Thus, in order to apply Theorem 1.1, one only needs to find the parameters α and β.

Theorem 4.1.

(a) *The spectrum of the operator* $-\Delta_\Gamma$ *is absolutely continuous and coincides with the interval* $l_d = [\gamma, \gamma + 4\sqrt{2d-1}]$, $\gamma = 2d - 2\sqrt{2d-1} \geq 0$.

(b) *The kernel of the parabolic semigroup* $\pi_\Gamma(t) = (e^{t\Delta_\Gamma})(t, e, e)$ *on the diagonal has the following asymptotic behavior at zero and infinity*

$$\pi_\Gamma(t) \to c_1 \text{ as } t \to 0, \ \pi_\Gamma(t) \sim c_2 \frac{e^{-\gamma t}}{t^{3/2}} \text{ as } t \to \infty. \tag{4.2}$$

Since the absolutely continuous spectrum of the operator $-\Delta_\Gamma$ is shifted (it starts from γ, not from zero), the natural question about the eigenvalues of the operator $-\Delta_\Gamma + V(g)$ is to estimate the number $N_\Gamma(V)$ of eigenvalues below the threshold γ. Obviously, $N_\Gamma(V)$ coincides with the number $N(V)$ of the negative eigenvalues of the operator $H_0 + V(g)$, where $H_0 = -\Delta_\Gamma - \gamma I$. The following statement immediately follows from (4.2) and Theorem 2.1

Theorem 4.2. *Estimate* (1.11) *with* $\alpha = 0, \beta = 3$ *is valid for operator* $H_0 = -\Delta_\Gamma - \gamma I$, *and*

$$N_\Gamma(V) \leq c(h)[n(h) + \sum_{g\in\Gamma: W(g)\leq h^{-1}} W(x)^{3/2}], \ n(h) = \#\{g \in \Gamma : W(g) > h^{-1}\}.$$

Proof of Theorem 4.1. Let us find the kernel $R_\lambda(g_1, g_2)$ of the resolvent $(\Delta_\Gamma - \lambda)^{-1}$. From the Γ-invariance it follows that $R_\lambda(g_1, g_2) = R_\lambda(e, g_1^{-1}g_2)$. Hence it is enough to determine $u_\lambda(g) = R_\lambda(e, g)$. This function satisfies the equation

$$\sum_{i\neq 0} u_\lambda(ga_i) - (2d + \lambda)u_\lambda(g) = -\delta_e(g), \tag{4.3}$$

where $\delta_e(g) = 1$ if $g = e$, $\delta_e(g) = 0$ if $g \neq e$. Since the equation above is preserved under permutations of the generators, the solution $u_\lambda(g)$ depends only on $m(g)$. Let $\psi_\lambda(m) = u_\lambda(g)$, $m = m(g)$. Obviously, if $g \neq e$, then $m(ga_i) = m(g) - 1$ for one of the elements a_i, $i \neq 0$, and $m(ga_i) = m(g) + 1$ for all other elements a_i, $i \neq 0$. Hence (4.3) implies

$$2d\psi_\lambda(1) - (2d + \lambda)\psi_\lambda(0) = -1, \tag{4.4}$$

$$\psi_\lambda(m-1) + (2d-1)\psi_\lambda(m+1) - (2d+\lambda)\psi_\lambda(m) = 0, \ m > 0.$$

Two linearly independent solutions of these equations have the form $\psi_\lambda(m) = \nu_\pm^m$, where $\nu_\pm$ are the roots of the equation

$$\nu^{-1} + (2d-1)\nu - (2d+\lambda) = 0.$$

Thus

$$\nu_\pm = \frac{2d + \lambda \pm \sqrt{(2d+\lambda)^2 - 4(2d-1)}}{2(2d-1)}.$$

The interval l_d was singled out as the set of real λ such that the discriminant above is not positive. Since $\nu_+\nu_- = 1/(2d-1)$, we have

$$|\nu_\pm| = \frac{1}{\sqrt{2d-1}} \text{ for } \lambda \in l_d; \ |\nu_+| > \frac{1}{\sqrt{2d-1}}, \ |\nu_-| < \frac{1}{\sqrt{2d-1}} \text{ for real } \lambda \notin l_d.$$

Now, if we take into account that the set $A_{m_0} = \{g \in \Gamma,\, m(g) = m_0\}$ has exactly $2d(2d-1)^{m_0-1}$ points, i.e., $\mu(A_{m_0}) = 2d(2d-1)^{m_0-1}$, we get that

$$\nu_-^{m(g)} \in L^2(\Gamma, \mu),\ \nu_+^{m(g)} \notin L^2(\Gamma, \mu) \text{ for real } \lambda \notin l_d, \tag{4.5}$$

and

$$\int_{\Gamma \cap \{g: m(g) \leq m_0\}} |\nu_\pm|^{2m(g)} \mu(dg) \sim m_0 \text{ as } m_0 \to \infty \quad \text{for } \lambda \notin l_d. \tag{4.6}$$

Relations (4.5) imply that $R \backslash l_d$ belongs to the resolvent set of the operator Δ_Γ and that $R_\lambda(e, g) = c\nu_-^{m(g)}$. Relation (4.6) implies that l_d belongs to the absolutely continuous spectrum of the operator Δ_Γ with functions $(\nu_+^{m(g)} - \nu_-^{m(g)})$ being the eigenfunctions of the continuous spectrum. Hence statement (a) is justified.

Note that the constant c in the formula for $R_\lambda(e, g)$ can be found from (4.4). This gives

$$R_\lambda(e, g) = \frac{1}{(2d + \lambda) - 2d\nu_-} \nu_-^{m(g)}.$$

Thus

$$R_\lambda(e, e) = \frac{1}{(2d + \lambda) - 2d\nu_-}.$$

Hence, for each $a > 0$,

$$\pi_\Gamma(t) = \frac{1}{2\pi} \int_{a - i\infty}^{a + i\infty} e^{\lambda t} R_\lambda(e, e) d\lambda = \frac{1}{2\pi} \int_{a - i\infty}^{a + i\infty} e^{\lambda t} \frac{d\lambda}{(2d + \lambda) - 2d\nu_-}.$$

The integrand here is analytic with branching points at the ends of the segment l_d, and the contour of integration can be bent into the left half-plane $\mathrm{Re}\lambda < 0$ and replaced by an arbitrary closed contour around l_d. This immediately implies the first relation of (4.2). The asymptotic behavior of the integral as $t \to \infty$ is defined by the singularity of the integrand at the point $-\gamma$ (the right end of l_d). Since the integrand there has the form $e^{\lambda t}[a + b\sqrt{\lambda + \gamma} + O(\lambda + \gamma)]$, $\lambda + \gamma \to 0$, this leads to the second relation of (4.2). □

4.2 General remark on left invariant diffusions on Lie groups. There are two standard ways to construct the Laplacian on a Lie group. A usual differential-geometric approach starts with the Lie algebra $\mathfrak{A}\Gamma$ on Γ, which can be considered either as the algebra of the first-order differential operators generated by the differentiations along the appropriate one-parameter subgroups of Γ, or simply as a tangent vector space $T\Gamma$ to Γ at the unit element I. The exponential mapping $\mathfrak{A}\Gamma \to \Gamma$ allows one to construct (at least locally) the general left invariant Laplacian $\triangle_\Gamma$ on Γ as the image of the differential operator $\sum_{ij} a_{ij} D_i D_j + \sum_i b_i D_i$ with constant coefficients on $\mathfrak{A}\Gamma$. The Riemannian metric ds^2 on Γ and the volume element dv can be defined now using the inverse matrix of the coefficients of the Laplacian $\triangle_\Gamma$. It is important to note that additional symmetry conditions are needed to determine $\triangle_\Gamma$ uniquely.

The central object in the probabilistic construction of the Laplacian (see, for instance, McKean [8]) is the Brownian motion g_t on Γ. We impose the symmetry condition $g_t \stackrel{\text{law}}{=} g_t^{-1}$. Since $\mathfrak{A}\Gamma$ is a linear space, one can define the usual Brownian motion b_t on $\mathfrak{A}\Gamma$ with the generator $\sum_{ij} a_{ij} D_i D_j + \sum_i b_i D_i$. The symmetry condition holds if $(I + db_t) \stackrel{\text{law}}{=} (I + db_t)^{-1}$. The process g_t (diffusion on Γ) is given (formally) by the stochastic multiplicative integral

$$g_t = \prod_{s=0}^{t} (I + db_s),$$

or (more rigorously) by the Ito's stochastic differential equation

$$dg_t = g_t db_t. \tag{4.7}$$

The Laplacian $\triangle_\Gamma$ is defined now as the generator of the diffusion:

$$\triangle_\Gamma f(g) = \lim_{\Delta t \to 0} \frac{Ef(g(I + b_{\Delta t})) - f(g)}{\Delta t}, \quad f \in C^2(\Gamma). \tag{4.8}$$

The Riemannian metric form is defined as above (by the inverse matrix of the coefficients of the Laplacian). We will use the probabilistic approach to construct the Laplacian in the examples below, since it allows us to easily incorporate the symmetry condition.

4.3 Heisenberg group $\Gamma = H^3$ of the upper triangular matrices

$$g = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, (x, y, z) \in R^3, \tag{4.9}$$

with units on the diagonal. We have

$$\mathfrak{A}\Gamma = \left\{ A = \begin{bmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix}, \quad (\alpha, \beta, \gamma) \in R^3 \right\},$$

$$e^A = \begin{bmatrix} 1 & \alpha & \gamma + \frac{\alpha\beta}{2} \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus $A \to \exp(A)$ is a one-to-one mapping of $\mathfrak{A}\Gamma$ onto Γ. Consider the following Brownian motion on $\mathfrak{A}\Gamma$:

$$b_t = \begin{bmatrix} 0 & u_t & \sigma w_t \\ 0 & 0 & v_t \\ 0 & 0 & 0 \end{bmatrix},$$

where σ is a constant and u_t, v_t, w_t are (standard) independent Wiener processes. Then equations (4.7), (4.8) imply that

$$(\Delta_\Gamma f)(x, y, z) = \frac{1}{2}[f_{xx} + f_{yy} + (\sigma^2 + x^2) f_{zz} + 2\sigma x f_{yz}].$$

This operator is self-adjoint with respect to the measure $dV = dxdydz$.

Denote by $p_\sigma(t,x,y,z)$ the transition density for the process g_t (fundamental solution of the parabolic equation $u_t = \Delta_\Gamma u$). Let $\pi_\sigma(t) = p_\sigma(t,0,0,0)$.

Theorem 4.3.

(1) *Function $\pi_\sigma(t)$ has the following asymptotic behavior at zero and infinity:*

$$\pi_\sigma(t) \sim \frac{c_0}{t^{3/2}}, \quad t \to 0; \quad \pi_\sigma(t) \sim \frac{c}{t^2}, \quad t \to \infty, \ c = p_0(1,0,0), \tag{4.10}$$

i.e., Theorem 1.1 *holds for operator $H = \Delta_\Gamma + V(x,y,z)$ with $\alpha = 3$, $\beta = 4$.*

(2) *Similar result (with $\alpha = 0, \beta = 4$) is valid for the Heisenberg discrete group $\Gamma = ZH^3$.*

4.4 Group $Aff\left(R^1\right)$ of affine transformations of the real line. This group of transformations $x \to ax + b$, $a > 0$, has a matrix representation:

$$\Gamma = Aff\left(R^1\right) = \{g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \quad a > 0, \quad (a,b) \in R^2\}.$$

We start with the Lie algebra for $\Gamma = Aff(R^1)$:

$$\mathfrak{A}\Gamma = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}, \quad (\alpha, \beta) \in R^2 \right\}$$

and the diffusion

$$b_t = \begin{bmatrix} w_t + \alpha t & v_t \\ 0 & 0 \end{bmatrix}$$

on $\mathfrak{A}\Gamma$, where (w_t, v_t) are independent Wiener processes. Consider the matrix-valued process $g_t = \begin{bmatrix} x_t & y_t \\ 0 & 1 \end{bmatrix}$, $g_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, on Γ satisfying the equation (4.7). Then $dx_t = x_t\left(dw_t + \alpha dt\right), \ dy_t = x_t dv_t$, i.e., (due to Ito's formula),

$$x_t = e^{w_t + \left(\alpha - \frac{1}{2}\right)t}, \quad y_t = \int_0^t x_s dv_s.$$

We impose the following symmetry condition: $\left(g_t\right)^{-1} \overset{\text{law}}{=} g_t$. One can show that it holds if $\alpha = \frac{1}{2}$, and that the generator of the process g_t has the form

$$\triangle_\Gamma f = \frac{x^2}{2}\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right] + \frac{x}{2}\frac{\partial f}{\partial x}.$$

Theorem 4.4.

(1) *Operator $\triangle_\Gamma$ is self-adjoint with respect to the measure $x^{-1}dxdy$. The function $\pi\left(t\right) = p\left(t,0,0\right)$ has the following behavior at zero and infinity:*

$$\pi\left(t\right) \sim \frac{c_0}{t}, \ t \to 0; \quad \pi\left(t\right) \sim \frac{C}{t^{3/2}}, \quad t \to \infty\ . \tag{4.11}$$

(2) *Let* $H = \triangle_\Gamma + V$, *where the negative part* $W = V_-$ *of the potential is bounded:* $W \le h^{-1}$. *From* (4.11) *and Theorem* 1.1 *it follows that*

$$N_0(V) \le C(h) \int_{-\infty}^{\infty} \int_0^{\infty} \frac{W^{3/2}(x,y)}{x} dxdy.$$

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ACL-homeomorphisms in the Plane

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To Vladimir Maz'ya on the occasion of his 70th birthday

Abstract. We study planar homeomorphisms $f : \Omega \subset \mathbb{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^2$, $f = (u, v)$, which are absolutely continuous on lines parallel to the axes (ACL) together with their inverse f^{-1}. The main result is that u and v have almost the same critical points. This generalizes a previous result ([8]) and extends investigation of ACL-solutions to non-trivial first-order systems of PDE's. The main ingredients are (N) and co-(N) properties of such mappings that we call ACL-homeomorphisms.

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1. Introduction

In [8] bisobolev homeomorphisms $f : \Omega \subset \mathbb{R}^n \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^n$ have been defined as those homeomorphisms such that f and f^{-1} belong to $W^{1,1}_{\text{loc}}$. In case $n = 2$, in [7] it is shown that bisobolev homeomorphisms have finite distortion, that is the differential $Df(x)$ is zero a.e. on the zero set of the Jacobian J_f. See also [3], Theorem 4.5.

Our aim here is to generalize this result to the case that f and f^{-1} are ACL-homeomorphisms in the plane, i.e., absolutely continuous on lines.

As a corollary we prove that the components of a ACL-homeomorphism f have the same critical points a.e., generalizing our previous result ([8]).

Notice that we cannot use coarea formula, which makes sense only for Sobolev maps. However, we will use the so-called co-(N) condition which holds for ACL mappings (see Section 3).

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The Gehring-Lehto Theorem guarantees that ACL-homeomorphisms are a.e. differentiable in the plane (Lemma 2.1) and we will use the fact that the Lusin condition (N) and the area formula hold in the set where f is differentiable and thus in particular, the image of the set of all critical point has zero measure (Sard's Lemma).

In this paper we also point out in the general setting of ACL-homeomorphisms $f = (u, v)$ the relevance of non-isotropic first-order degenerate elliptic systems of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla v = A(x) \nabla u.$$

Here $A = (a_{ij}) : \ \Omega \to \mathbb{R}^{2\times 2}$ is a measurable symmetric matrix-valued function such that

$$^{t}A(x) = A(x) \qquad \qquad \det A(x) = 1 \qquad \text{a.e.}$$

under the ellipticity bounds

$$\frac{|\xi|^2}{K(x)} \le \langle A(x)\xi, \xi \rangle \le K(x)|\xi|^2 \qquad \text{a.e. } x$$

for all $\xi \in \mathbb{R}^2$ and for a Borel function $K : \ \Omega \to [1, +\infty)$.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Fix $t \in \mathbb{R}$. Since $\mathbb{R} \times \{t\}$ is a copy of $\mathbb{R}$, the Hausdorff measure on it can be identified with the Lebesgue measure and we can write dz instead of $d\mathcal{H}^1(z)$.

Given a 2×2 matrix D we define the norm $|D|$ as the supremum of $|D\xi|$ over all vectors $\xi \in \mathbb{R}^2$ of unit Euclidean norm.

A mapping $f : \Omega \to \mathbb{R}^2$ is said to satisfy the Lusin condition (N) on the measurable set $E \subset \Omega$ if $|f(A)| = 0$ for every $A \subset E$ such that $|A| = 0$.

A function $u = u(x_1, x_2)$ continuous in Ω is said *absolutely continuous on lines* in Ω if for every rectangle

$$]a, b[\times]c, d[\subset\subset \Omega$$

u is absolutely continuous as a function of the real variable x_1 on a.e. segment $I_{x_2} =]a, b[\times\{x_2\}$ and as a function of x_2 on a.e. segment $\{x_1\}\times]c, d[$.

It is well known (see [11], Lemma III.3.1 and [6]) that a continuous function $u : \Omega \to \mathbb{R}$ which is absolutely continuous on lines (ACL for short) in Ω, possesses finite partial derivatives a.e. in Ω.

In the following we will assume that $f = (u, v) : \ \Omega \subset \mathbb{R}^2 \to \Omega' \subset \mathbb{R}^2$ is a homeomorphism with u and v ACL together with the components of the inverse f^{-1} and call such a mapping an ACL-*homeomorphism*.

Lemma 2.1. (*Gehring-Lehto*) *If $f:\Omega \xrightarrow{\text{onto}} \Omega'$ is an ACL-homeomorphism, then for $i,j \in \{1,2\}$ the partial derivatives $\frac{\partial f^{(i)}}{\partial x_j}(x)$ exist and are finite for a.e. $x \in \Omega$. Moreover f is a.e. differentiable in Ω in the classical sense.*

For the proof see [11] Theorem III.3.1.

From Lemma 2.1 it follows that if f is an ACL-homeomorphism, then either $J_f(x) \geq 0$ or $J_f(x) \leq 0$ a.e. We will assume from now on $J_f(x) \geq 0$ a.e.

If E and F are measurable subsets of $\mathbb{R}^2$ we will write

$$E = F \quad a.e. \quad \text{if} \quad |(E \cup F) \setminus (E \cap F)| = 0.$$

In [8] the following definition was introduced

Definition 2.2. Let Ω and Ω' be bounded domains in $\mathbb{R}^n$ and $1 \leq p \leq \infty$. A homeomorphism $f:\Omega \xrightarrow{\text{onto}} \Omega'$ is said to be a $W^{1,p}$-bisobolev map if f belongs to the Sobolev space $W^{1,p}_{\text{loc}}(\Omega;\Omega')$ and its inverse $g = f^{-1}$ belongs to $W^{1,p}_{\text{loc}}(\Omega';\Omega)$. If $p = 1$ we say that f is bisobolev. If $p = \infty$ the $W^{1,\infty}$-bisobolev maps are bilipschitz.

It is well known that $W^{1,1}_{\text{loc}}$ functions are absolutely continuous on lines (see [11], Sect. III.6.1), whence bisobolev maps are particular ACL-homeomorphisms. A sufficient condition for a homeomorphism $f:\Omega \xrightarrow{\text{onto}} \Omega'$ to be a bisobolev map is given by the following ([7])

Proposition 2.3. *If the homeomorphism $f:\Omega \xrightarrow{\text{onto}} \Omega'$ belongs to $W^{1,1}_{\text{loc}}(\Omega;\Omega')$ and its Jacobian is strictly positive a.e., then $f^{-1} \in W^{1,1}_{\text{loc}}(\Omega';\Omega)$, hence f is bisobolev.*

Notice that in general, for a ACL-homeomorphism, it may happen that the zero set $\mathcal{Z}_f$ of the Jacobian $J_f = \det Df$

$$\mathcal{Z}_f = \{x \in \mathcal{D}_f : \; J_f(x) = 0\}$$

has positive measure. Hence, by Sard's Lemma, which guarantees that

$$|f(\mathcal{Z}_f)| = 0,$$

it follows that the (N)-property of Lusin may be violated (see [16]).

Therefore, in general

$$\int_B J_f(x)\,dx \leq |f(B)|$$

where the equality may be unattainable on certain Borel sets B ([11], p. 137).

However the (N)-property takes place if $f \in W^{1,2}_{\text{loc}}$ ([11], p. 158). Observe then that, for $W^{1,2}$-homeomorphisms $f = (u,v)$, i.e., if f and f^{-1} are of class $W^{1,2}_{\text{loc}}$, Sard's Lemma and the (N)-condition on f^{-1} imply that the critical sets of u and v have both zero measure.

Let us finally adopt a recent terminology; we say that an homeomorphism such that $|Df(x)| = 0$ on the zero set of J_f has *finite distortion.*

3. Properties of ACL-homeomorphisms

In what follows, we will denote by

$$p_i : \ x \in \mathbb{R}^2 \to \mathbb{H}_i = \{x \in \mathbb{R}^2 : x_i = 0\}$$

the orthogonal projection to $\mathbb{H}_i$, i.e.,

$$p_1(x) = (0, x_2)$$
$$p_2(x) = (x_1, 0).$$

We denote by p^j the projection to the jth coordinate $p^j(x) = x_j$.

The following co-(N) property, which is an adaptation of coarea formula to ACL-mappings, can be proved exactly as Theorem 4.4 in [3].

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $f : \Omega \stackrel{\text{onto}}{\longrightarrow} \Omega' \subset \mathbb{R}^2$ be an ACL-homeomorphism. Then, for any measurable set $E \subset \Omega$ such that $|E| = 0$ we have*

$$\int_{\mathbb{H}_i} \mathcal{H}^1(E \cap (p_i \circ f)^{-1}(z))\, dz = 0.$$

The following Lusin N property on lines is crucial in the proof of Proposition 3.1 and has independent interest

Proposition 3.2. *Let $g :]a, b[\times]c, d[\longrightarrow \mathbb{R}^2$ be an ACL-homeomorphism. Then, for a.e. $y_2 \in]c, d[$ if $A \subset]a, b[\times\{y_2\}$ verifies $\mathcal{H}^1(A) = 0$, then*

$$\mathcal{H}^1(g(A)) = 0.$$

The main result of this paper is the following

Theorem 3.3. *Let $f : \Omega \subset \mathbb{R}^2 \stackrel{\text{onto}}{\longrightarrow} \Omega' \subset \mathbb{R}^2$ be an ACL-homeomorphism. Then the differential matrix of its inverse $Df^{-1}(y)$ vanishes a.e. on the zero set of the Jacobian determinant $J_{f^{-1}}$.*

Proof. Suppose by contradiction that there exists a set $\bar{A} \subset f(\Omega)$ with positive Lebesgue measure such that

$$J_{f^{-1}}(y) = 0 \qquad \text{and} \qquad |Df^{-1}(y)| > 0 \qquad\qquad \forall y \in \bar{A}.$$

Since f^{-1} is ACL, we may assume without loss of generality that f^{-1} is absolutely continuous on all lines parallel to coordinate axes that intersect A.

If we represent f^{-1} by mean of its components $f^{-1}(y) = (s(y), t(y))$ we can suppose

$$\left|\frac{\partial s}{\partial y_1}(y)\right| > 0 \qquad\qquad \forall y \in \bar{A} \tag{3.1}$$

or

$$\left|\frac{\partial t}{\partial y_2}(y)\right| > 0 \qquad\qquad \forall y \in \bar{A}. \tag{3.2}$$

Otherwise if (3.1) and (3.2) fail both, then we can replace f^{-1} with $g(y) = (t(y), s(y))$ and then we have necessarily

$$\left|\frac{\partial t}{\partial y_1}(y)\right| > 0 \qquad \forall y \in \bar{A}$$

or

$$\left|\frac{\partial s}{\partial y_2}(y)\right| > 0 \qquad \forall y \in \bar{A}.$$

So we assume (3.1). The proof in case (3.2) is similar.

Since in view of Lemma 2.1 f^{-1} is differentiable a.e. in Ω', there exists a Borel set $A \subset \bar{A}$ such that $|A| > 0$ and f^{-1} is differentiable at every point of A. Hence we are legitimate to apply the area formula on A to find that

$$|f^{-1}(A)| = \int_{\Omega} \chi_{f^{-1}(A)}(x)\, dx = \int_{f(\Omega)} \chi_A(y) J_{f^{-1}}(y)\, dy = 0 \tag{3.3}$$

where we used that $J_{f^{-1}}(y) \equiv 0$ on A. Notice that (3.3) is a version of Sard's Lemma. Since f^{-1} is an ACL-homeomorphism, f^{-1} is of bounded variation on lines. Hence $(p_i \circ f)^{-1}(z)$ is a rectifiable curve for $i = 1, 2$. By co-(N) property we can write

$$\int_{\mathbb{H}_1} \mathcal{H}^1(f^{-1}(A) \cap f^{-1}(p_1^{-1}(z)))\, dz = 0.$$

Thus the curve $(p_1 \circ f)^{-1}(z)$ has zero one-dimensional measure for a.e. $z \in \mathbb{H}_1$, and its two projections to the axis have zero one-dimensional measure as well. Therefore

$$\mathcal{H}^1(p_j(f^{-1}(A) \cap (p_1 \circ f)^{-1}(y))) = 0 = \mathcal{H}^1(p_j(\{x \in f^{-1}(A) : p_1 \circ f(x) = z\}))$$

for a.e. $z \in \mathbb{H}_1$ and for $j = 1, 2$. Using the Fubini Theorem, we have

$$|A| = \int_{\mathbb{H}_1} |A \cap p_1^{-1}(z)|\, dz > 0. \tag{3.4}$$

Hence there exists $z_0 \in \mathbb{H}_1$ such that

$$\mathcal{H}^1(A \cap p_1^{-1}(z_0)) > 0$$

but

$$\mathcal{H}^1(p_j(\{x \in f^{-1}(A) : p_1 \circ f(x) = z_0\})) = 0. \tag{3.5}$$

Applying the one-dimensional area formula for the absolutely continuous function $s(\cdot, z_o) : \tau \in p^1(A) \to s(z_0 + \tau \mathbf{e}_1)$ we have

$$\begin{aligned} 0 &< \int_{A \cap p_1^{-1}(z_0)} \left|\frac{\partial s}{\partial y_1}(y_1, z_0)\right| d\mathcal{H}^1(y) = \int_{\mathbb{R}} N(s, A \cap p_1^{-1}(z_0), \sigma)\, d\sigma \\ &= \int_{f^{-1}(A) \cap (p_1 \circ f)^{-1}(z_0)} N(s, A \cap p_1^{-1}(z_0), \sigma)\, d\sigma \end{aligned} \tag{3.6}$$

where $N(s, A \cap p_1^{-1}(z_0), \sigma)$ is the number of preimages of σ under s in $A \cap p_1^{-1}(z_0)$. The last integral in (3.6) is zero by (3.5) and this is a contradiction. □

By symmetry, we can rewrite Theorem 3.3 as follows

Theorem 3.4. *Let $f : \Omega \subset \mathbb{R}^2 \overset{\text{onto}}{\longrightarrow} \Omega' \subset \mathbb{R}^2$ be an ACL-homeomorphism. Then its differential matrix $Df(x)$ vanishes a.e. on the zero set of the Jacobian determinant J_f.*

According to well-known terminology, an homeomorphism $g : \Omega \to \Omega'$ is said of *finite distortion* if $Dg(x)$ vanishes a.e. on the zero set of $J_g(x)$. Theorems 3.3 and 3.4 claim that ACL-homeomorphisms have finite distortion.

As a consequence we have the following

Corollary 3.5. *Let $f = (u, v) : \Omega \subset \mathbb{R}^2 \overset{\text{onto}}{\longrightarrow} \Omega' \subset \mathbb{R}^2$ be an ACL-homeomorphism. Then the components of f have the same critical points, i.e.,*

$$\{x \in \Omega : \ \nabla u = 0\} = \{x \in \Omega : \ \nabla v = 0\} \qquad \text{a.e.}$$

Proof. By Theorem 3.4, we have that

$$\{x \in \Omega : \ J_f(x) = 0\} \subset \{x \in \Omega : \ |Df(x)| = 0\} \qquad \text{a.e.}$$

Hence

$$\{\, \nabla u(x) = 0\} \subset \{\, J_f(x) = 0\} \subset \{\, |Df(x)| = 0\} \subset \{\, \nabla v(x) = 0\} \qquad \text{a.e.}$$

The other inclusion is completely analogous. □

Remark 3.6. Notice that for ACL-homeomorphisms the two conditions

$$J_f(x) = 0 \quad \Rightarrow \quad |Df(x)| = 0 \qquad \text{a.e.} \tag{3.7}$$

and

$$\{x \in \Omega : \ \nabla u = 0\} = \{x \in \Omega : \ \nabla v = 0\} \qquad \text{a.e.} \tag{3.8}$$

are equivalent. The equivalence between (3.7) and (3.8) is false when f is not one-to-one. For example if we choose $f = (u, u)$ then (3.8) holds but (3.7) does not hold.

This means that in order to prove (3.8) ⇒ (3.7) one needs properties of f^{-1}.

Remark 3.7. Corollary 3.5 fails if we relax the AC-condition on $f = f(x_1, x_2)$ with respect to one of the two variables into BV-condition. We give an example of a homeomorphism

$$f = f(x_1, x_2) = (h(x_1), x_2) \tag{3.9}$$

where $f = f(\cdot, x_2)$ is a function of bounded variation for a.e. x_2 but the two components u and v of f do not have the same critical points a.e. ([7]).

Define $c : (0, 1) \to (0, 1)$ as the ternary Cantor function, which is an increasing homeomorphism, and set $g(t) = t + c(t)$, so that $h = g^{-1}$ maps $(0, 2)$ homeomorphically onto $(0, 1)$, $h \in W^{1,\infty}_{\text{loc}}$ but $g \notin W^{1,1}_{\text{loc}}$. Hence the inverse homeomorphism of f given by (3.9)

$$f^{-1}(y_1, y_2) = (g(y_1), y_2)$$

is not ACL, but only BVL (of bounded variation on lines). If we set $f(x) = (u(x), v(x))$ we see that

$$\begin{aligned} |\nabla u(x)| &= 0 \qquad \text{a.e.} \\ |\nabla v(x)| &= 1 \qquad \text{a.e.} \end{aligned}$$

hence the critical sets of u and v disagree a.e.

Remark 3.8. The result of Corollary 3.5 does not hold for $n > 2$. In [8] an example is given of a bisobolev map $f : \Omega \subset \mathbb{R}^3 \to \Omega' \subset \mathbb{R}^3$ whose three components do not share the same critical set (a.e.)

4. ACL-homeomorphisms and first-order systems of PDE's

The interplay between planar elliptic PDE's and Function Theory has been known since pioneering work of C.B. Morrey [14]. In this section we will obtain a far reaching generalization of some results from [14]. We will show that for each ACL-homeomorphism $f : \Omega \subset \mathbb{R}^2 \overset{\text{onto}}{\longrightarrow} \Omega' \subset \mathbb{R}^2$, $f = (u, v)$, the functions u and v are very weak solutions to a non-trivial degenerate elliptic system

$$\begin{cases} \quad v_{x_2}(x) = a_{11}(x)u_{x_1}(x) + a_{12}(x)u_{x_2}(x) \\ - \quad v_{x_1}(x) = a_{12}(x)u_{x_1}(x) + a_{22}(x)u_{x_2}(x) \end{cases} \quad \text{for a.e. } x \in \Omega. \tag{4.1}$$

where $A = (a_{ij}) : \Omega \to \mathbb{R}^{2\times 2}$ is a measurable matrix-valued function such that $a_{ij} = a_{ji}$, $\det A(x) = 1$ and for a.e. $x \in \Omega$

$$\frac{|\xi|^2}{K(x)} \le \langle A(x)\xi, \xi\rangle \le K(x)|\xi|^2 \qquad \forall \xi \in \mathbb{R}^2 \tag{4.2}$$

for a Borel function $K : \Omega \to [1, +\infty)$. Here it is crucial that u and v have the same set of critical points. In fact we have the following

Theorem 4.1. *To each ACL-homeomorphism* $f : \Omega \to \Omega'$, $f = (u, v)$, *there corresponds a measurable function* $A = A(x)$ *valued in symmetric matrices with* $\det A(x) = 1$ *such that for a.e.* $x \in \Omega$ *and for all* $\xi \in \mathbb{R}^2$ *we have* (4.2) *for a Borel function* $K(x) : \Omega \to [1, +\infty)$ *and the components of* f *are very weak solutions to the system* (4.1) *with finite energy, i.e.,*

$$\int_\Omega \langle A(x)\nabla u, \nabla u\rangle < \infty \text{ and } \int_\Omega \langle A(x)\nabla v, \nabla v\rangle < \infty\,.$$

Proof. In this proof we follow some ideas of [9] where the case of $W^{1,2}$ solutions was studied. Let us set

$$\begin{aligned} a_{11}(x) &= \frac{v_{x_2}^2(x) + u_{x_2}^2(x)}{J_f(x)}, \quad a_{22}(x) = \frac{v_{x_1}^2(x) + u_{x_1}^2(x)}{J_f(x)}, \\ a_{12}(x) &= a_{21}(x) = -\frac{v_{x_1}(x)v_{x_2}(x) + u_{x_1}(x)u_{x_2}(x)}{J_f(x)} \end{aligned} \tag{4.3}$$

if $x \in \Omega$ is such that $J_f(x) \neq 0$. For the points $x \in \Omega$ such that $J_f(x) = 0$, we set

$$a_{ij}(x) = \delta_{ij}.$$

It can be easily checked that the matrix $A = \mathcal{A}_f = (a_{ij})$ is symmetric, $\det A(x) = 1$ and (4.2) holds for a suitable Borel function $K(x)$.

The equations in (4.1) can be verified by a direct computation for every point such that $J_f(x) \neq 0$. If $J_f(x) = 0$ then $|Df(x)| = 0$ by Theorem 3.4 and therefore (4.1) is clearly satisfied. From (4.1) we see that $\langle A(x)\nabla v, \nabla v\rangle = J_f(x)$ and therefore we can use area formula to conclude

$$\int_\Omega \langle A(x)\nabla v(x), \nabla v(x)\rangle \, dx = \int_\Omega J_f(x)\, dx \leq |f(\Omega)| < \infty.$$

Analogously we can check that u is a very weak solution with finite energy ([8]).

□

Remark 4.2. If we introduce the distortion function of the ACL-homeomorphism $f : \Omega \to \Omega'$

$$K_f(x) = \begin{cases} \frac{|Df(x)|^2}{J_f(x)} & \text{for } J_f(x) > 0 \\ 1 & \text{for } J_f(x) = 0. \end{cases} \tag{4.4}$$

We immediately see that $\mathcal{A}_f(x)$ satisfies (4.2) with $K = K_f$.

Remark 4.3. Let us note that previous device for generating linear elliptic equations from ACL-mappings is reminiscent of De Giorgi's construction of linear elliptic systems with discontinuous coefficients and unbounded solutions [4], [18]. In fact, after the famous solution to the problem of Hölder continuity of solutions of elliptic equations with measurable coefficients ([5], [15], see also [13]) De Giorgi in [4] gave a counterexample in case of systems, starting with a function and constructing an adapted system. In case $f = (u, v)$ is bisobolev, the components $u, v \in W^{1,1}_{\mathrm{loc}}$ are very weak solutions to the divergence type equations

$$\mathrm{div}\mathcal{A}_f(x)\nabla u = 0 \qquad\qquad \mathrm{div}\mathcal{A}_f(x)\nabla v = 0.$$

In other words, f is a solution to the linear elliptic degenerate system

$$\mathrm{Div}(\mathcal{A}_f(x)\, {}^tDf) = 0$$

where for $M \in L^1(\Omega; \mathbb{R}^{2\times 2})$ we define

$$(\mathrm{Div}M(x))_i = \sum_{j=1}^{2} \frac{\partial M_{ji}}{\partial x_j}(x).$$

Remark 4.4. In case of homeomorphisms with bounded or exponentially integrable distortion K, that is there exists $\lambda > 0$ such that $\int_\Omega \exp\left(\frac{K(x)}{\lambda}\right) dx < \infty$, the properties of the operator $f \to \mathcal{A}_f$ have been studied by [19] and [2] respectively (see also [17]). It turns out that if $f_j \to f$ locally uniformly then $\mathcal{A}_{f_j} \to \mathcal{A}_f$ in the sense of Γ-convergence ([12]).

5. Mappings with integrable distortion

The study of properties of bisobolev mappings $f : \Omega \subset \mathbb{R}^2 \overset{\text{onto}}{\longrightarrow} \Omega' \subset \mathbb{R}^2$ such that the distortion function K_f belongs to $L^1(\Omega)$ was performed in [1], [7], [10] and the outcome was that the inverse f^{-1} has $W^{1,2}(\Omega';\Omega)$ regularity.

Our aim here is to generalize this result to ACL-homeomorphisms. Following closely [7] we will prove the following for sake of completeness.

Theorem 5.1. *Let $f : \Omega \subset \mathbb{R}^2 \overset{\text{onto}}{\longrightarrow} \Omega' \subset \mathbb{R}^2$ be an ACL-homeomorphism such that $K_f \in L^1(\Omega)$. Then $f^{-1} \in W^{1,2}_{\text{loc}}(\Omega';\Omega)$.*

Proof. We claim that there exists a Borel set $A \subset f(\Omega)$ such that f^{-1} is differentiable on A with $J_{f^{-1}}(y) > 0$ for all $y \in A$ so that $|Df^{-1}(y)| = 0$ a.e. on $f(\Omega) \setminus A$. In fact, f^{-1} has finite distortion by Theorem 3.3 hence we have $J_{f^{-1}}(y) = 0 \Rightarrow |Df^{-1}(y)|^2 = 0$ and therefore we will restrict our care to the set $\tilde{A} = \{y : J_{f^{-1}}(y) > 0\}$.

Let us introduce the following subsets of $\tilde{A}$

$$\mathcal{R} = \{y \in \tilde{A} :\ f \text{ is differentiable at } f^{-1}(y) \text{ and } J_f(f^{-1}(y)) > 0\}$$
$$\mathcal{Z} = \{y \in \tilde{A} :\ f \text{ is differentiable at } f^{-1}(y) \text{ and } J_f(f^{-1}(y)) = 0\}$$
$$\mathcal{E} = \{y \in \tilde{A} :\ f \text{ is not differentiable at } f^{-1}(y)\}.$$

By Sard's Lemma we have that $|\mathcal{Z}| = 0$. By the area formula we deduce

$$\int_{\mathcal{E}} J_{f^{-1}}(y)\, dy \le |f^{-1}(\mathcal{E})| = 0.$$

Since $J_{f^{-1}} > 0$ on $\mathcal{E}$ this implies that $|\mathcal{E}| = 0$. So we can take $A = \mathcal{R}$.

Since f is differentiable at every point of $f^{-1}(A)$ and $J_f > 0$ we deduce that f^{-1} differentiable in A. We recall also that $J_f(x) = 0$ a.e. in $\Omega \setminus f^{-1}(A)$, hence by finite distortion for f, $|Df(x)| = 0$ a.e. in $\Omega \setminus f^{-1}(A)$. Finally we have

$$\begin{aligned}\int_{f(\Omega)} |Df^{-1}(y)|^2\, dy &= \int_A |Df^{-1}(y)|^2\, dy \\ &= \int_{f^{-1}(A)} \frac{|Df^{-1}(f(x))|^2}{J_{f^{-1}}(f(x))}\, dx \\ &= \int_{f^{-1}(A)} |(Df(x))^{-1}|^2 J_f(x)\, dx \\ &= \int_{f^{-1}(A)} \frac{|Df(x)|^2}{J_f(x)}\, dx \\ &\le \int_\Omega K_f(x)\, dx.\end{aligned} \tag{5.1}$$

□

The following result seems interesting as well

Proposition 5.2. *Let f be an ACL-homeomorphism $f : \Omega \to \Omega'$ such that $f^{-1} \in W^{1,2}_{\text{loc}}(\Omega';\Omega)$. Then $J_f(x) > 0$ a.e. $x \in \Omega$.*

Proof. By the Gehring-Lehto Theorem we know that f is a.e. differentiable. Let us define the Borel set

$$E = \{x \in \Omega : \ f \text{ is differentiable at } x \text{ and } J_f(x) = 0\}$$

We claim that $|E| = 0$. Assume the contrary $|E| > 0$ then necessarily $|f(E)| > 0$, otherwise $f^{-1}(f(E))$ would have zero measure, because f^{-1} verifies the (N) property ([11], III.6.1). Actually $|f(E)| > 0$ would imply that f^{-1} is not differentiable on the positive set $f(E)$: a contradiction. (This is a version of Sard's Lemma.) So we have $|E| = 0$ and $J_f(x) > 0$ for a.e. $x \in \Omega$. □

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Crack Problems for Composite Structures

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Dedicated to Vladimir Maz'ya on the occasion of his 70th birthday

Abstract. We investigate three-dimensional interface crack problems (ICP) for metallic-piezoelectric composite bodies with regard to thermal effects. We give a mathematical formulation of the physical problem when the metallic and piezoelectric bodies are bonded along some proper parts of their boundaries where interface cracks occur. By potential methods the ICP is reduced to an equivalent strongly elliptic system of pseudodifferential equations on manifolds with boundary. We study the solvability of this system in different function spaces and prove uniqueness and existence theorems for the original ICP. We analyze the regularity properties of the corresponding thermo-mechanical and electric fields near the crack edges and near the curves where the boundary conditions change. In particular, we characterize the stress singularity exponents, which essentially depend on the material parameters, and show that they can be explicitly calculated with the help of the principal homogeneous symbol matrices of the corresponding pseudodifferential operators.

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1. Introduction

In this paper we investigate a mathematical model describing the interaction of the elastic, thermal and electric fields in a three-dimensional composite consisting of a piezoelectric (ceramic) matrix and metallic inclusions (electrodes) bonded along some proper parts of their boundaries where interface cracks occur. Modern industrial and technological processes apply widely such type composite materials. Therefore investigation of the mathematical models for such composite materials and analysis of the corresponding thermo-mechanical and electric fields became very actual and important for fundamental research and practical applications (for details see [6], [8], [9] and the references therein).

Here we consider a general three-dimensional interface crack problem (ICP) for an anisotropic piezoelectric-metallic composite with regard to thermal effects and perform a rigorous mathematical analysis by the potential method.

In the piezoceramic part the unknown field is represented by a 5-component vector (three components of the displacement vector, the electric potential function and the temperature), while in the metallic part the unknown field is described by a 4-component vector (three components of the displacement vector and the temperature).

As it is well known from the classical mathematical physics and the classical elasticity theory, in general, solutions to crack type and mixed boundary value problems have singularities near the crack edges and near the lines where the types of boundary conditions change, regardless of the smoothness of given boundary data. The same effect can be observed in the case of our interface crack problem, namely, singularities of electric, thermal and stress fields appear near the crack edges and near lines, where the boundary conditions change and where the interfaces intersect the exterior boundary (the so-called *exceptional curves*).

We apply the generalized potential method developed in the references [3], [5], [7], and reduce the ICP to the equivalent system of pseudodifferential equations on a proper part of the boundary of the composed body. We analyze the solvability of the resulting boundary-integral equations in Sobolev-Slobodetski (W^s_p), Bessel potential (H^s_p), and Besov ($B^s_{p,t}$) spaces and prove the corresponding uniqueness and existence theorems for the original ICP. Moreover, our main goal is a detailed theoretical investigation of regularity properties of thermo-mechanical and electric fields near the exceptional curves and qualitative description of their singularities. In particular, we show the global C^α-regularity results with some $\alpha \in (\,0,\ \frac{1}{2}\,]$ depending on the eigenvalues of a matrix which is explicitly constructed with the help of the homogeneous symbol matrices of the corresponding pseudodifferential operators. These eigenvalues depend on the material parameters, in general, and actually they define the singularity exponents for the first-order derivatives of solutions. Consequently, the important result is that the stress singularity exponents essentially depend on material parameters. We recall that for interior cracks the stress singularity exponents do not depend on the material parameters and equal to -0.5 (see, e.g., [3], [4], [9]).

2. Formulation of the problem and uniqueness result

2.1. Geometrical description of the composite configuration. Let $\Omega^{(m)}$ and Ω be bounded disjoint domains of the three-dimensional Euclidean space $\mathbb{R}^3$ with C^∞-smooth boundaries $\partial\Omega^{(m)}$ and $\partial\Omega$, respectively. Moreover, let $\partial\Omega$ and $\partial\Omega^{(m)}$ have a nonempty, simply connected intersection $\overline{\Gamma^{(m)}}$ with a positive measure, i.e., $\partial\Omega \cap \partial\Omega^{(m)} = \overline{\Gamma^{(m)}}$, $\operatorname{meas}\Gamma^{(m)} > 0$. From now on $\Gamma^{(m)}$ will be referred to as an interface surface. Throughout the paper n and $\nu = n^{(m)}$ stand for the outward unit normal vectors on $\partial\Omega$ and on $\partial\Omega^{(m)}$, respectively. Evidently, $n(x) = -\nu(x)$ for $x \in \Gamma^{(m)}$.

Further, let $\overline{\Gamma^{(m)}} = \overline{\Gamma_T^{(m)}} \cup \overline{\Gamma_C^{(m)}}$, where $\Gamma_C^{(m)}$ is an open, simply connected proper part of $\Gamma^{(m)}$. Moreover, $\Gamma_T^{(m)} \cap \Gamma_C^{(m)} = \varnothing$ and $\partial\Gamma^{(m)} \cap \overline{\Gamma_C^{(m)}} = \varnothing$.

We set $S_N^{(m)} := \partial\Omega^{(m)} \setminus \overline{\Gamma^{(m)}}$ and $S^* := \partial\Omega \setminus \overline{\Gamma^{(m)}}$. Further, we denote by S_D some open, nonempty, proper sub-manifold of S^* and let $S_N := S^* \setminus \overline{S_D}$. Thus, we have the following dissections of the boundary surfaces (see Figure 1)

$$\partial\Omega = \overline{\Gamma_T^{(m)}} \cup \overline{\Gamma_C^{(m)}} \cup \overline{S_N} \cup \overline{S_D}, \quad \partial\Omega^{(m)} = \overline{\Gamma_T^{(m)}} \cup \overline{\Gamma_C^{(m)}} \cup \overline{S_N^{(m)}}.$$

Throughout the paper, for simplicity, we assume that $\partial\Omega^{(m)}$, $\partial\Omega$, $\partial S_N^{(m)}$, $\partial\Gamma_T^{(m)}$, $\partial\Gamma_C^{(m)}$, ∂S_D, ∂S_N are C^∞-smooth and $\partial\Omega^{(m)} \cap \overline{S_D} = \varnothing$.

Let Ω be filled by an anisotropic homogeneous piezoelectric medium (ceramic matrix) and $\Omega^{(m)}$ be occupied by an isotropic or anisotropic homogeneous elastic medium (metallic inclusion). These two bodies interact to each other along the interface $\Gamma^{(m)}$ where the interface crack $\Gamma_C^{(m)}$ occurs. Moreover, it is assumed that the composed body is fixed along the sub-surface S_D (the Dirichlet part of the boundary), while the sub-manifolds $S_N^{(m)}$ and S_N are the Neumann parts of the boundary. In the metallic domain $\Omega^{(m)}$ we have a four-dimensional thermoelastic field described by the displacement vector $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)})^\top$ and temperature distribution function $u_4^{(m)} = \vartheta^{(m)}$, while in the piezoelectric domain Ω we have a five-dimensional physical field described by the displacement vector $u = (u_1, u_2, u_3)^\top$, temperature distribution function $u_4 = \vartheta$ and the electric potential $u_5 = \varphi$. We set $U^{(m)} := (u^{(m)}, \vartheta^{(m)})^\top$ and $U := (u, \vartheta, \varphi)^\top$. The superscript $(\cdot)^\top$ denotes transposition operation. Throughout the paper the summation over the repeated indices is meant from 1 to 3, unless stated otherwise.

2.2. Field equations in $\Omega^{(m)}$. Components of the mechanical thermo-stress tensor read as $\sigma_{ij}^{(m)} = c_{ijlk}^{(m)} \partial_l u_k^{(m)} - \gamma_{ij}^{(m)} \vartheta^{(m)}$, $i, j = 1, 2, 3$, while the components of the heat flux vector read as $q_i^{(m)} = -\varkappa_{il}^{(m)} \partial_l \vartheta^{(m)}$, $i = 1, 2, 3$. Here and in what follows we employ the notation $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$.

The vector $U^{(m)} = (u^{(m)}, \vartheta^{(m)})^\top$ solves the system of differential equations in the metallic domain $\Omega^{(m)}$ (for details see [7], [8])

$$\begin{aligned} &c_{ijlk}^{(m)} \partial_i \partial_l u_k^{(m)} - \varrho^{(m)} \tau^2 u_j^{(m)} - \gamma_{ij}^{(m)} \partial_i \vartheta^{(m)} = 0, \quad j = 1, 2, 3, \\ &-\tau T_0 \gamma_{il}^{(m)} \partial_l u_i^{(m)} + \varkappa_{il}^{(m)} \partial_i \partial_l \vartheta^{(m)} - \tau \alpha^{(m)} \vartheta^{(m)} = 0, \end{aligned} \tag{2.1}$$

or in matrix form

$$A^{(m)}(\partial, \tau) U^{(m)}(x) = 0, \tag{2.2}$$

where $A^{(m)}(\partial, \tau)$ is the nonselfadjoint 4×4 matrix differential operator generated by equations (2.1).

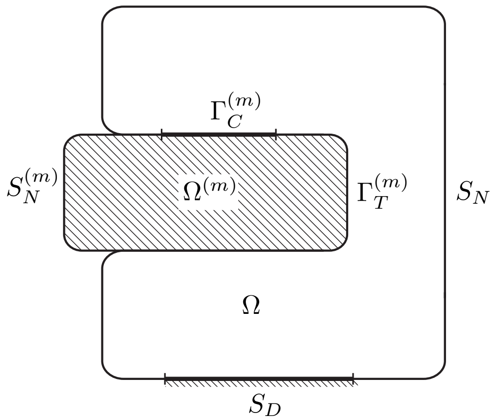

FIGURE 1. Composite body

Here $c_{ijkl}^{(m)}$ are the elastic constants, $\varrho^{(m)}$ is the mass density, $\varkappa_{kj}^{(m)}$ are the thermal conductivity constants, $\gamma_{kj}^{(m)}$ are the thermal strain constants, T_0 is the initial reference temperature, $\alpha^{(m)} = \varrho^{(m)}\,\widetilde{c}^{(m)}$ with $\widetilde{c}^{(m)}$ as the specific heat per unit mass, $\tau = \sigma + i\,\omega$ is a complex parameter.

Constants involved in the above equations satisfy the symmetry conditions:

$$c_{ijkl}^{(m)} = c_{jikl}^{(m)} = c_{klij}^{(m)}, \quad \gamma_{ij}^{(m)} = \gamma_{ji}^{(m)}, \quad \varkappa_{ij}^{(m)} = \varkappa_{ji}^{(m)}, \quad i,j,k,l = 1,2,3,$$

and for all $\xi_{ij} = \xi_{ji} \in \mathbb{R}$, $i,j = 1,2,3$, and $\xi = (\xi_j, \xi_j, \xi_j)^\top \in \mathbb{R}^3$ there are positive constants $C_0^{(m)}$ and $C_1^{(m)}$, such that

$$c_{ijkl}^{(m)}\,\xi_{ij}\,\xi_{kl} \geq C_0^{(m)}\,\xi_{ij}\,\xi_{ij}, \quad \varkappa_{ij}^{(m)}\xi_i\xi_j \geq C_1^{(m)}\,|\xi|^2.$$

Further, we introduce the following generalized thermo-stress operator

$$\mathcal{T}^{(m)}(\partial,\nu) = [\,\mathcal{T}_{jk}^{(m)}(\partial,\nu)\,]_{4\times 4},$$

where

$$\begin{aligned} &\mathcal{T}_{jk}^{(m)}(\partial,\nu) = c_{ijlk}^{(m)}\,\nu_i\,\partial_l, \;\; \mathcal{T}_{j4}^{(m)}(\partial,\nu) = -\gamma_{ij}^{(m)}\,\nu_i, \\ &\mathcal{T}_{4k}^{(m)}(\partial,\nu) = 0, \;\; \mathcal{T}_{44}^{(m)}(\partial,\nu) = \varkappa_{il}^{(m)}\,\nu_i\,\partial_l, \quad j,k = 1,2,3. \end{aligned}$$

For a vector $U^{(m)} = (u^{(m)}, \vartheta^{(m)})^\top$ we have

$$\mathcal{T}^{(m)}\,U^{(m)} = (\,\sigma_{i1}^{(m)}\,\nu_i,\; \sigma_{i2}^{(m)}\,\nu_i,\; \sigma_{i3}^{(m)}\,\nu_i,\; \varkappa_{il}^{(m)}\,\nu_i\,\partial_l\,)^\top. \tag{2.3}$$

The components of the vector $\mathcal{T}^{(m)}\,U^{(m)}$ given by (2.3) have the physical sense: the first three components correspond to the thermo-mechanical stress vector acting on a surface element with a normal $\nu = (\nu_1, \nu_2, \nu_3)$, while the forth one is the normal component of the heat flux vector (with opposite sign).

2.3. Field equations in Ω. The constitutive relations in the piezoelectric domain Ω read as

$$\sigma_{ij} = c_{ijkl}\,\partial_l u_k + e_{lij}\,\partial_l\varphi - \gamma_{ij}\,\vartheta,$$
$$D_j = e_{jkl}\,\partial_l u_k - \varepsilon_{jl}\,\partial_l\varphi + g_j\,\vartheta,\ \ q_j = -\varkappa_{jl}\,\partial_l\vartheta,\quad i,j=1,2,3,$$

where σ_{kj} is the mechanical stress tensor in the theory of thermoelectroelasticity, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector, $E = (E_1, E_2, E_3)^\top := -\mathrm{grad}\,\varphi$ is the electric field vector, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, ϱ is the mass density, c_{ijkl} are the elastic constants, e_{kij} are the piezoelectric constants, ε_{kj} are the dielectric (permittivity) constants, γ_{kj} are thermal strain constants, $\varkappa_{kj}$ are thermal conductivity constants, $\alpha := \varrho\,\widetilde{c}$ with $\widetilde{c}$ as the specific heat per unit mass, g_i are pyroelectric constants characterizing the relation between thermodynamic processes and piezoelectric effects.

The constants involved in these equations satisfy the symmetry conditions:

$$c_{ijkl}=c_{jikl}=c_{klij},\ \ e_{ijk}=e_{ikj},\ \varepsilon_{ij}=\varepsilon_{ji},\ \gamma_{ij}=\gamma_{ji},\ \varkappa_{ij}=\varkappa_{ji},\ i,j,k,l=1,2,3.$$

Moreover, from physical considerations it follows that for all $\xi_{ij} = \xi_{ji} \in \mathbb{R}$, $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{C}^3$, and $\zeta \in \mathbb{C}$ (see, e.g., [8]):

$$c_{ijkl}\,\xi_{ij}\,\xi_{kl} \geq C_0\,\xi_{ij}\,\xi_{ij},\ \ \varepsilon_{ij}\,\eta_i\,\overline{\eta}_j \geq C_1\,|\eta|^2\,,\ \ \varkappa_{ij}\,\eta_i\,\overline{\eta}_j \geq C_2\,|\eta|^2\,,$$
$$\varepsilon_{ij}\,\eta_i\,\overline{\eta}_j + \frac{\alpha}{T_0}\,|\,\zeta|^2 - 2\,\Re\,(\,\zeta\,g_l\,\overline{\eta}_l\,) \geq C_3(\,|\zeta|^2 + |\eta|^2\,)$$

where C_0, C_1, C_2 and C_3 are positive constants.

The vector $U = (u, \vartheta, \varphi)^\top$ solves the following system of differential equations in the piezoelectric domain Ω (for details see [7], [8])

$$\begin{aligned}
&c_{ijlk}\,\partial_i\,\partial_l\,u_k - \varrho\,\tau^2\,u_j - \gamma_{ij}\,\partial_i\vartheta + e_{lij}\,\partial_l\partial_i\varphi = 0,\quad j=1,2,3,\\
&-\tau\,T_0\,\gamma_{il}\,\partial_l u_i + \varkappa_{il}\,\partial_i\,\partial_l\vartheta - \tau\,\alpha\,\vartheta + \tau\,T_0\,g_i\,\partial_i\varphi = 0,\\
&-e_{ikl}\,\partial_i\partial_l u_k - g_i\,\partial_i\vartheta + \varepsilon_{il}\,\partial_i\partial_l\varphi = 0,
\end{aligned}\tag{2.4}$$

or in matrix form

$$A(\partial,\tau)\,U(x) = 0 \ \text{ in } \ \Omega,\tag{2.5}$$

where $A(\partial,\tau)$ is a nonselfadjoint 5×5 matrix differential operator generated by equations (2.4).

Let us introduce the generalized stress operator

$$\begin{aligned}
&\mathcal{T}(\partial,n) = \big[\,\mathcal{T}_{jk}(\partial,n)\,\big]_{5\times 5},\\
&\mathcal{T}_{jk}(\partial,n) = c_{ijlk}\,n_i\,\partial_l,\quad \mathcal{T}_{j4}(\partial,n) = -\gamma_{ij}\,n_i,\quad \mathcal{T}_{j5}(\partial,n) = e_{lij}\,n_i\,\partial_l,\\
&\mathcal{T}_{4k}(\partial,n) = 0,\quad \mathcal{T}_{44}(\partial,n) = \varkappa_{il}\,n_i\,\partial_l,\quad \mathcal{T}_{45}(\partial,n) = 0,\\
&\mathcal{T}_{5k}(\partial,n) = -e_{ikl}\,n_i\,\partial_l,\ \mathcal{T}_{54}(\partial,n) = -g_i\,n_i,\ \mathcal{T}_{55}(\partial,n) = \varepsilon_{il}\,n_i\,\partial_l,\ \ j,k=1,2,3.
\end{aligned}$$

For a vector $U = (u, \varphi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial,n)\,U = (\,\sigma_{i1}\,n_i,\ \sigma_{i2}\,n_i,\ \sigma_{i3}\,n_i,\ -q_i\,n_i,\ -D_i\,n_i\,)^\top.\tag{2.6}$$

The components of the vector $\mathcal{T}\,U$ given by (2.6) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermoelectroelasticity, the forth and fifth ones are the normal components of the heat flux vector and the electric displacement vector (with opposite sign), respectively.

2.4. Formulation of the interface crack problem. Let us introduce some notation. Throughout the paper the symbol $\{\,\cdot\,\}^{+}$ denotes the interior one-sided limit on $\partial\Omega$ (respectively $\partial\Omega^{(m)}$) from Ω (respectively $\Omega^{(m)}$). Similarly, $\{\,\cdot\,\}^{-}$ denotes the exterior one-sided limit on $\partial\Omega$ (respectively $\partial\Omega^{(m)}$) from the exterior of Ω (respectively $\Omega^{(m)}$).

By L_p, W_p^r, H_p^s, and $B_{p,q}^s$ (with $r\geq 0$, $s\in\mathbb{R}$, $1<p<\infty$, $1\leq q\leq\infty$) we denote the well-known Lebesgue, Sobolev-Slobodetski, Bessel potential, and Besov function spaces, respectively (see, e.g., [11]). Recall that $H_2^r=W_2^r=B_{2,2}^r$, $H_2^s=B_{2,2}^s$, $W_p^t=B_{p,p}^t$, and $H_p^k=W_p^k$, for any $r\geq 0$, for any $s\in\mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

Let $\mathcal{M}_0$ be a smooth surface without boundary. For a smooth sub-manifold $\mathcal{M}\subset\mathcal{M}_0$ we denote by $\widetilde{H}_p^s(\mathcal{M})$ and $\widetilde{B}_{p,q}^s(\mathcal{M})$ the subspaces of $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$\widetilde{H}_p^s(\mathcal{M})=\{g:g\in H_p^s(\mathcal{M}_0),\ \operatorname{supp}g\subset\overline{\mathcal{M}}\},$$
$$\widetilde{B}_{p,q}^s(\mathcal{M})=\{g:g\in B_{p,q}^s(\mathcal{M}_0),\ \operatorname{supp}g\subset\overline{\mathcal{M}}\},$$

while $H_p^s(\mathcal{M})$ and $B_{p,q}^s(\mathcal{M})$ denote the spaces of restrictions on $\mathcal{M}$ of functions from $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$H_p^s(\mathcal{M})=\{r_{\mathcal{M}}f:\ f\in H_p^s(\mathcal{M}_0)\},\quad B_{p,q}^s(\mathcal{M})=\{r_{\mathcal{M}}f\ :\ f\in B_{p,q}^s(\mathcal{M}_0)\},$$

where $r_{\mathcal{M}}$ is the restriction operator on $\mathcal{M}$.

Now we formulate mathematically the mixed interface crack problems when the crack gap is assumed to be thermally insulated and electrically nonpermeable.

Problem (ICP-A): Find solution vector functions $U^{(m)}\in[W_p^1(\Omega^{(m)})]^4$ and $U\in[W_p^1(\Omega)]^5$ with $1<p<\infty$ of the systems (2.1) and (2.4), respectively, satisfying

(i) *the boundary conditions*:

$$r_{S_N^{(m)}}\big\{[\mathcal{T}^{(m)}(\partial,\nu)\,U^{(m)}]_j\big\}^{+}=Q_j^{(m)}\quad\text{on}\quad S_N^{(m)},\ \ j=\overline{1,4},\tag{2.7}$$
$$r_{S_N}\{[\mathcal{T}(\partial,n)\,U]_k\}^{+}=Q_k\quad\text{on}\quad S_N,\ \ k=\overline{1,5},\tag{2.8}$$
$$r_{S_D}\{u_k\}^{+}=f_k\quad\text{on}\quad S_D,\ \ k=\overline{1,5},\tag{2.9}$$
$$r_{\Gamma_T^{(m)}}\{u_5\}^{+}=f_5^{(m)}\quad\text{on}\quad\Gamma_T^{(m)},\tag{2.10}$$

(ii) *the transmission conditions* ($j=\overline{1,4}$):

$$r_{\Gamma_T^{(m)}}\{u_j\}^{+}-r_{\Gamma_T^{(m)}}\{u_j^{(m)}\}^{+}=f_j^{(m)}\ \text{on}\ \Gamma_T^{(m)},\tag{2.11}$$
$$r_{\Gamma_T^{(m)}}\big\{[\mathcal{T}(\partial,n)U]_j\big\}^{+}+r_{\Gamma_T^{(m)}}\big\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_j\big\}^{+}=F_j^{(m)}\ \text{on}\ \Gamma_T^{(m)},\tag{2.12}$$

(iii) *the interface crack conditions*:

$$r_{\Gamma_C^{(m)}}\{[\mathcal{T}^{(m)}(\partial,\nu)\,U^{(m)}]_j\}^+=\widetilde{Q}_j^{(m)} \quad \text{on} \quad \Gamma_C^{(m)},\ \ j=\overline{1,4}, \tag{2.13}$$

$$r_{\Gamma_C^{(m)}}\{[\mathcal{T}(\partial,n)\,U]_k\}^+=\widetilde{Q}_k \quad \text{on} \quad \Gamma_C^{(m)},\ \ k=\overline{1,5}, \tag{2.14}$$

where $n=-\nu$ on $\Gamma^{(m)}$,

$$\begin{aligned}
&Q_k\in B_{p,p}^{-\frac{1}{p}}(S_N),\quad Q_j^{(m)}\in B_{p,p}^{-\frac{1}{p}}(S_N^{(m)}),\quad f_k\in B_{p,p}^{1-\frac{1}{p}}(S_D),\\
&f_k^{(m)}\in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T^{(m)}),\quad F_j^{(m)}\in B_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)}),\quad \widetilde{Q}_j^{(m)}\in B_{p,p}^{-\frac{1}{p}}(\Gamma_C^{(m)}),\\
&\widetilde{Q}_k\in B_{p,p}^{-\frac{1}{p}}(\Gamma_C^{(m)}),\quad k=\overline{1,5},\quad j=\overline{1,4}.
\end{aligned} \tag{2.15}$$

2.5. Green's formulas and uniqueness theorem. Green's identities for vector functions

$$U^{(m)}=(u_1^{(m)},\dots,u_4^{(m)})^\top,\ V^{(m)}=(v_1^{(m)},\dots,v_4^{(m)})^\top\in[W_2^1(\Omega^{(m)})]^4,$$
$$U=(u_1,\dots,u_5)^\top,\ V=(v_1,\dots,,v_5)^\top\in[W_2^1(\Omega)]^5$$

with $A^{(m)}U^{(m)}\in[L_2(\Omega^{(m)})]^4$ and $AU\in[L_2(\Omega^{(m)})]^5$, read as (cf. [7])

$$\begin{aligned}
&\Big\langle\{\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}\}^+,\{V^{(m)}\}^+\Big\rangle_{\partial\Omega^{(m)}}=\int_{\Omega^{(m)}}A^{(m)}(\partial,\tau)U^{(m)}\cdot V^{(m)}\,dx\\
&+\int_{\Omega^{(m)}}\Big[E^{(m)}(u^{(m)},\overline{v^{(m)}})+\varrho^{(m)}\,\tau^2\,u_j^{(m)}\overline{v_j^{(m)}}+\varkappa_{jl}^{(m)}\,\partial_j u_4^{(m)}\,\overline{\partial_l v_4^{(m)}}\\
&+\tau\,\alpha^{(m)}\,u_4^{(m)}\,\overline{v_4^{(m)}}+\gamma_{jl}^{(m)}\left(\tau\,T_0\partial_j u_l^{(m)}\,\overline{v_4^{(m)}}-u_4^{(m)}\,\overline{\partial_j v_l^{(m)}}\right)\Big]\,dx,
\end{aligned}$$

$$\begin{aligned}
\big\langle\{\mathcal{T}(\partial,n)U\}^+,\{V\}^+\big\rangle_{\partial\Omega}&=\int_\Omega A(\partial,\tau)\,U\cdot V\,dx+\int_\Omega\Big[E(u,\overline{v})+\varrho\,\tau^2\,u_j\overline{v_j}\\
&+\gamma_{jl}\,(\,\tau\,T_0\,\partial_j u_l\,\overline{v_4}-u_4\,\overline{\partial_j v_l}\,)+\varkappa_{jl}\,\partial_j u_4\,\overline{\partial_l v_4}+\tau\,\alpha\,u_4\,\overline{v_4}+\varepsilon_{jl}\,\partial_j u_5\,\overline{\partial_l v_5}\\
&+e_{lij}\,(\partial_l u_5\,\overline{\partial_i v_j}-\partial_i u_j\,\overline{\partial_l v_5})-g_l\,(\,\tau\,T_0\,\partial_l u_5\,\overline{v_4}+u_4\,\overline{\partial_l v_5})\Big]\,dx,
\end{aligned}$$

where $E^{(m)}(u^{(m)},\overline{v^{(m)}})=c_{ijlk}^{(m)}\,\partial_i u_j^{(m)}\,\overline{\partial_l v_k^{(m)}}$ and $E(u,\overline{v})=c_{ijlk}\,\partial_i u_j\,\overline{\partial_l v_k}$. Here the central dot denotes the scalar product in $\mathbb{C}^N$, $N=4,5$, and $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega^{(m)}}$ (respectively $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$) denotes the duality between the function spaces $[B_{2,2}^{-\frac{1}{2}}(\partial\Omega^{(m)})]^4$ and $[B_{2,2}^{\frac{1}{2}}(\partial\Omega^{(m)})]^4$ (respectively $[B_{2,2,}^{-\frac{1}{2}}(\partial\Omega)]^5$ and $[B_{2,2,}^{\frac{1}{2}}(\partial\Omega)]^5$) which extends the usual L_2 scalar product.

Now we formulate the following uniqueness result.

Theorem 2.1. *Let $\tau=\sigma+i\omega$ with $\sigma>0$. The interface crack problem (ICP-A) has at most one solution in the space $[W_2^1(\Omega^{(m)})]^4\times[W_2^1(\Omega)]^5$, provided the measure of S_D is positive.*

Proof. It follows from the above Green's formulas and the positive definiteness properties of material parameters described in Subsections 2.2 and 2.3. □

3. Existence and regularity results for Problem (ICP-A)

3.1. Layer potentials. Denote by $\Psi^{(m)}(\cdot,\tau)=\big[\Psi^{(m)}_{kj}(\cdot,\tau)\big]_{4\times 4}$ and $\Psi(\cdot,\tau)=\big[\Psi_{kj}(\cdot,\tau)\big]_{5\times 5}$ the fundamental matrix-functions of the operators $A^{(m)}(\partial,\tau)$ and $A(\partial,\tau)$ and introduce the single layer potentials (see [1], [7]):

$$V^{(m)}_\tau(h^{(m)})(x):=\int_{\partial\Omega^{(m)}}\Psi^{(m)}(x-y,\tau)\,h^{(m)}(y)\,d_yS,$$

$$V_\tau(h)(x):=\int_{\partial\Omega}\Psi(x-y,\tau)\,h(y)\,d_yS,$$

where $h^{(m)}=(h^{(m)}_1,h^{(m)}_2,h^{(m)}_3,h^{(m)}_4)^\top$ and $h=(h_1,h_2,h_3,h_4,h_5)^\top$ are densities of the potentials. For the boundary integral (pseudodifferential) operators generated by the layer potentials we will employ the following notation:

$$\mathcal{H}^{(m)}_\tau(h^{(m)})(x):=\int\limits_{\partial\Omega^{(m)}}\Psi^{(m)}(x-y,\tau)\,h^{(m)}(y)\,d_yS,\ x\in\partial\Omega^{(m)},$$

$$\mathcal{K}^{(m)}_\tau(h^{(m)})(x):=\int\limits_{\partial\Omega^{(m)}}\big[\mathcal{T}^{(m)}(\partial_x,\nu(x))\Psi^{(m)}(x-y,\tau)\big]h^{(m)}(y)d_yS,\ x\in\partial\Omega^{(m)},$$

$$\mathcal{H}_\tau(h)(x):=\int\limits_{\partial\Omega}\Psi(x-y,\tau)\,h(y)\,d_yS,\ x\in\partial\Omega,$$

$$\mathcal{K}_\tau(h)(x):=\int\limits_{\partial\Omega}\big[\mathcal{T}(\partial_x,n(x))\Psi(x-y,\tau)\big]\,h(y)\,d_yS,\quad x\in\partial\Omega.$$

The layer boundary operators $\mathcal{H}^{(m)}_\tau$ and $\mathcal{H}_\tau$ are pseudodifferential operators of order -1, while the operators $\mathcal{K}^{(m)}_\tau$ and $\mathcal{K}_\tau$ are singular integral operators, i.e., pseudodifferential operators of order 0 (for details see the [1], [7]).

In [7] it is shown that if $\Re\,\tau=\sigma>0$ and $1<p<\infty$, then

(i) an arbitrary solution $U^{(m)}\in[W^1_p(\Omega^{(m)})]^4$ to the homogeneous equation (2.2) can be uniquely represented by the single layer potential

$$U^{(m)}(x)=V^{(m)}_\tau\big(\big[\mathcal{P}^{(m)}_\tau\big]^{-1}\chi^{(m)}\big)(x),\quad x\in\Omega^{(m)},$$

where

$$\mathcal{P}^{(m)}_\tau:=-2^{-1}I_4+\mathcal{K}^{(m)}_\tau\quad\text{and}\quad\chi^{(m)}=\big\{\mathcal{T}^{(m)}U^{(m)}\big\}^+\in\big[B^{-\frac{1}{p}}_{p,p}(\partial\Omega^{(m)})\big]^4.$$

Moreover, $\mathcal{P}^{(m)}_\tau:\big[B^s_{p,t}(\partial\Omega^{(m)})\big]^4\to\big[B^s_{p,t}(\partial\Omega^{(m)})\big]^4$ is an invertible operator for $s\in\mathbb{R}$, $1<p<\infty$, and $1\le q\le\infty$;

(ii) an arbitrary solution $U\in[W^1_p(\Omega)]^5$ to the homogeneous equation (2.5) can be uniquely represented by the single layer potential $U(x)=V_\tau\big(\mathcal{P}^{-1}_\tau\chi\big)(x)$, where $\chi=\big\{\mathcal{T}U\big\}^++\beta\,\big\{U\big\}^+\in\big[B^{-\frac{1}{p}}_{p,p}(\partial\Omega)\big]^5$ and $\mathcal{P}_\tau:=-2^{-1}I_5+\mathcal{K}_\tau+\beta\,\mathcal{H}_\tau$ with

β being a smooth real-valued scalar function which does not vanish identically and

$$\beta \geq 0 \quad \text{and} \quad \operatorname{supp}\beta \subset S_D. \tag{3.1}$$

The properties of the single layer potentials and the corresponding boundary operators are described by the following theorems (see [7]). Below we assume that $1 < p < \infty$, $1 \leq t \leq \infty$, and $s \in \mathbb{R}$.

Theorem 3.1. *The operators* $V_\tau^{(m)} : \big[\, B_{p,p}^{s}(\partial\Omega^{(m)})\,\big]^4 \to \big[\, H_p^{s+1+\frac{1}{p}}(\Omega^{(m)})\,\big]^4$ *and* $V_\tau : \big[\, B_{p,p}^{s}(\partial\Omega)\,\big]^5 \to \big[\, H_p^{s+1+\frac{1}{p}}(\Omega)\,\big]^5$ *are continuous.*

Theorem 3.2. *Let* $h^{(m)} \in \big[B_{p,t}^{-\frac{1}{p}}(\partial\Omega^{(m)})\big]^4$ *and* $h \in \big[B_{p,t}^{-\frac{1}{p}}(\partial\Omega)\big]^5$*. Then*

$$\begin{aligned}
&\big\{V_\tau^{(m)}(h^{(m)})\big\}^+ = \big\{V_\tau^{(m)}(h^{(m)})\big\}^- = \mathcal{H}_\tau^{(m)}\, h^{(m)} \quad on \quad \partial\Omega^{(m)},\\
&\big\{\mathcal{T}^{(m)}(\partial,\nu)V_\tau^{(m)}(h^{(m)})\big\}^\pm = \big[\, \mp\, 2^{-1}I_4 + \mathcal{K}_\tau^{(m)}\,\big]\, h^{(m)} \quad on \quad \partial\Omega^{(m)},\\
&\big\{V_\tau(h)\big\}^+ = \big\{V_\tau(h)\big\}^- = \mathcal{H}_\tau\, h \quad on \quad \partial\Omega,\\
&\big\{\mathcal{T}(\partial,n)V_\tau(h)\big\}^\pm = \big[\, \mp\, 2^{-1}I_5 + \mathcal{K}_\tau\,\big]\, h \quad on \quad \partial\Omega,
\end{aligned}$$

where I_k *stands for the* $k \times k$ *unit matrix.*

Theorem 3.3. *The operators*

$$\begin{aligned}
\mathcal{H}_\tau^{(m)} &: \big[B_{p,t}^{s}(\partial\Omega^{(m)})\big]^4 \to \big[B_{p,t}^{s+1}(\partial\Omega^{(m)})\big]^4,\\
\mathcal{K}_\tau^{(m)} &: \big[B_{p,t}^{s}(\partial\Omega^{(m)})\big]^4 \to \big[B_{p,t}^{s}(\partial\Omega^{(m)})\big]^4,\\
\mathcal{H}_\tau &: \big[B_{p,t}^{s}(\partial\Omega)\big]^5 \to \big[B_{p,t}^{s+1}(\partial\Omega)\big]^5,\\
\mathcal{K}_\tau &: \big[B_{p,t}^{s}(\partial\Omega)\big]^5 \to \big[B_{p,t}^{s}(\partial\Omega)\big]^5,
\end{aligned}$$

are continuous. Moreover, the operators $\mathcal{H}_\tau^{(m)}$, $\mathcal{H}_\tau$ *and* $-\frac{1}{2}\, I_4 + \mathcal{K}_\tau^{(m)}$ *are invertible if* $\sigma > 0$, *while the operator* $-\frac{1}{2}\, I_5 + \mathcal{K}_\tau$ *is Fredholm with zero index for any* $\tau \in \mathbb{C}$.

3.2. Reduction to boundary equations. Here we derive the equivalent boundary integral formulation of the interface crack problem (ICP-A). Keeping in mind (2.15), let

$$G := \begin{cases} Q & \text{on} \quad S_N,\\ \widetilde{Q} & \text{on} \quad \Gamma_C^{(m)}, \end{cases} \qquad G^{(m)} := \begin{cases} Q^{(m)} & \text{on} \quad S_N^{(m)},\\ \widetilde{Q}^{(m)} & \text{on} \quad \Gamma_C^{(m)}, \end{cases}$$

$$G \in [\, B_{p,p}^{-\frac{1}{p}}(S_N \cup \Gamma_C^{(m)})\,]^5, \qquad G^{(m)} \in [\, B_{p,p}^{-\frac{1}{p}}(S_N^{(m)} \cup \Gamma_C^{(m)})\,]^4.$$

Let $G_0 = (G_{01}, \dots, G_{05})^\top$ and $G_0^{(m)} = (G_{01}^{(m)}, \dots, G_{04}^{(m)})^\top$ be some fixed extensions of the vector functions G and $G^{(m)}$ respectively onto $\partial\Omega$ and $\partial\Omega^{(m)}$ preserving the space, i.e., $G_0 \in [\, B_{p,p}^{-\frac{1}{p}}(\partial\Omega)\,]^5$ and $G_0^{(m)} \in [\, B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)})]^4$. It is evident that arbitrary extensions of the same vector functions can be represented then as $G^* = G_0 + \psi + h$ and $G^{(m)*} = G_0^{(m)} + h^{(m)}$, where

$$\psi := (\psi_1, \dots, \psi_5)^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D) \,\big]^5,$$
$$h := (h_1, \dots, h_5)^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)}) \,\big]^5,$$
$$h^{(m)} := (h_1^{(m)}, \dots, h_4^{(m)})^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)}) \,\big]^4,$$

are arbitrary vector functions.

We develop here the so-called indirect boundary integral equations method and look for a solution pair $(U^{(m)}, U)$ of the interface crack problem (ICP-A) in the form of single layer potentials,

$$U^{(m)} = V_\tau^{(m)}\big(\big[\mathcal{P}_\tau^{(m)}\big]^{-1} \big[G_0^{(m)} + h^{(m)} \big] \big) \quad \text{in} \quad \Omega^{(m)}, \tag{3.2}$$
$$U = V_\tau \left(\mathcal{P}_\tau^{-1} \left[G_0 + \psi + h \right] \right) \quad \text{in} \quad \Omega, \tag{3.3}$$

where $\mathcal{P}_\tau^{(m)}$ and $\mathcal{P}_\tau$ are defined above.

Due to (3.1) and the properties of the single layer potentials we can easily verify that the homogeneous differential equations (2.2) and (2.5), the boundary conditions (2.7)–(2.8) and crack conditions (2.13)–(2.14) are satisfied automatically.

The remaining boundary and transmission conditions (2.9)–(2.12) lead to the simultaneous pseudodifferential equations with respect to the unknown vector functions ψ, h and $h^{(m)}$

$$r_{S_D} \big[\mathcal{H}_\tau \mathcal{P}_\tau^{-1} \big(\psi + h \big) \big]_k = \widetilde{f}_k \quad \text{on} \quad S_D, \quad k = \overline{1,5}, \tag{3.4}$$
$$r_{\Gamma_T^{(m)}} \big[\mathcal{H}_\tau \mathcal{P}_\tau^{-1} \big(\psi + h \big) \big]_5 = \widetilde{f}_5^{(m)} \quad \text{on} \quad \Gamma_T^{(m)}, \tag{3.5}$$
$$r_{\Gamma_T^{(m)}} \big[\mathcal{H}_\tau \mathcal{P}_\tau^{-1} \big(\psi + h \big) \big]_j - r_{\Gamma_T^{(m)}} \big[\mathcal{H}_\tau^{(m)} \big[\mathcal{P}_\tau^{(m)}\big]^{-1} h^{(m)} \big]_j = \widetilde{f}_j^{(m)} \quad \text{on} \quad \Gamma_T^{(m)},\ j = \overline{1,4}, \tag{3.6}$$
$$r_{\Gamma_T^{(m)}} h_j^{(m)} + r_{\Gamma_T^{(m)}} h_j = \widetilde{F}_j^{(m)} \quad \text{on} \quad \Gamma_T^{(m)},\ j = \overline{1,4}. \tag{3.7}$$

where

$$\widetilde{f}_k := f_k - r_{S_D} [\mathcal{H}_\tau \mathcal{P}_\tau^{-1} G_0]_k \in B_{p,p}^{1-\frac{1}{p}}(S_D), \quad k = \overline{1,5},$$
$$\widetilde{f}_5^{(m)} := f_5^{(m)} - r_{\Gamma_T^{(m)}} \big[\mathcal{H}_\tau \mathcal{P}_\tau^{-1} G_0 \big]_5 \in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T^{(m)}),$$
$$\widetilde{f}_j^{(m)} := f_j^{(m)} + r_{\Gamma_T^{(m)}} \big[\mathcal{H}_\tau^{(m)} \big[\mathcal{P}_\tau^{(m)}\big]^{-1} G_0^{(m)} \big]_j - r_{\Gamma_T^{(m)}} \big[\mathcal{H}_\tau \mathcal{P}_\tau^{-1} G_0 \big]_j \in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T^{(m)}), \quad j = \overline{1,4},$$
$$\widetilde{F}_j^{(m)} := F_j^{(m)} - r_{\Gamma_T^{(m)}} G_{0j} - r_{\Gamma_T^{(m)}} G_{0j}^{(m)} \in r_{\Gamma_T^{(m)}} \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)}),\ j = \overline{1,4}. \tag{3.8}$$

The last inclusions are the *compatibility conditions* for Problem (ICP-A). In what follows we assume that $\widetilde{F}_j^{(m)}$ are extended from $\Gamma_T^{(m)}$ to $\partial\Omega^{(m)} \cup \partial\Omega$ by zero, i.e., $\widetilde{F}_j^{(m)} \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)})$, $j=\overline{1,3}$.

Let us introduce the Steklov-Poincaré type 5×5 matrix pseudodifferential operators

$$\mathcal{A}_\tau := \mathcal{H}_\tau\,\mathcal{P}_\tau^{-1}, \qquad \mathcal{B}_\tau^{(m)} := \begin{bmatrix} [\mathcal{H}_\tau^{(m)}\,[\mathcal{P}_\tau^{(m)}]^{-1}]_{4\times 4} & [\,0\,]_{4\times 1} \\ [\,0\,]_{1\times 4} & [\,0\,]_{1\times 1} \end{bmatrix}_{5\times 5},$$

and rewrite equations (3.4)–(3.7) as

$$r_{S_D}\,\mathcal{A}_\tau\big(\psi + h\big) = \widetilde{f} \quad \text{on} \quad S_D, \tag{3.9}$$

$$r_{\Gamma_T^{(m)}}\,\mathcal{A}_\tau\big(\psi + h\big) + r_{\Gamma_T^{(m)}}\,\mathcal{B}_\tau^{(m)} h = \widetilde{g}^{(m)} \quad \text{on} \quad \Gamma_T^{(m)}, \tag{3.10}$$

$$r_{\Gamma_T^{(m)}}\,h_j + r_{\Gamma_T^{(m)}}\,h_j^{(m)} = \widetilde{F}_j^{(m)} \quad \text{on} \quad \Gamma_T^{(m)},\ j=\overline{1,4}, \tag{3.11}$$

where $\widetilde{f} \in [B_{p,p}^{1-\frac{1}{p}}(S_D)]^5$, $\widetilde{F}^{(m)} \in [\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)})]^4$, and $\widetilde{g}^{(m)} \in [B_{p,p}^{1-\frac{1}{p}}(\Gamma_T^{(m)})]^5$ with $\widetilde{g}_j^{(m)} := \widetilde{f}_j^{(m)} + r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau^{(m)}\,\big[\mathcal{P}_\tau^{(m)}\big]^{-1}\,\widetilde{F}^{(m)}\big]_j$, $j=\overline{1,4}$, $\widetilde{g}_5^{(m)} = \widetilde{f}_5^{(m)}$.

We note here that since the unknown vector function h is supported on $\Gamma_T^{(m)}$, the operator $\mathcal{B}_\tau^{(m)}h$ is defined correctly provided h is extended by zero on $S_N^{(m)} \cup \Gamma_C^{(m)}$. For this extended vector function we will keep the same notation h. So, actually, in what follows we can assume that h is a vector function defined on $\partial\Omega \cup \partial\Omega^{(m)}$ and is supported on $\Gamma_T^{(m)}$.

3.3. Existence theorems and regularity of solutions. Here we show that the system of pseudodifferential equations (3.9)–(3.11) is uniquely solvable in appropriate function spaces. To this end, let us put

$$\mathcal{N}_\tau^{(A)} := \begin{bmatrix} r_{S_D}\,\mathcal{A}_\tau & r_{S_D}\,\mathcal{A}_\tau & r_{S_D}\,[\,0\,]_{5\times 4} \\ r_{\Gamma_T^{(m)}}\,\mathcal{A}_\tau & r_{\Gamma_T^{(m)}}[\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] & r_{\Gamma_T^{(m)}}\,[\,0\,]_{5\times 4} \\ r_{\Gamma_T^{(m)}}\,[\,0\,]_{4\times 5} & r_{\Gamma_T^{(m)}}\,I_{4\times 5} & r_{\Gamma_T^{(m)}}\,I_4 \end{bmatrix}_{14\times 14} \tag{3.12}$$

with $I_{4\times 5} := [\,I_4\ ,\ [0]_{4\times 1}\,]_{4\times 5}$.

Further, let $\Phi := (\psi,\, h,\, h^{(m)})^\top$, $Y := (\widetilde{f},\, \widetilde{g}^{(m)},\, \widetilde{F}^{(m)})^\top$, and

$$\begin{aligned}
\mathbf{X}_p^s &:= [\widetilde{B}_{p,p}^s(S_D)]^5 \times [\widetilde{B}_{p,p}^s(\Gamma_T^{(m)})]^5 \times [\widetilde{B}_{p,p}^s(\Gamma_T^{(m)})]^4,\\
\mathbf{Y}_p^s &:= [B_{p,p}^{s+1}(S_D)]^5 \times [B_{p,p}^{s+1}(\Gamma_T^{(m)})]^5 \times [\widetilde{B}_{p,p}^s(\Gamma_T^{(m)})]^4,\\
\mathbf{X}_{p,q}^s &:= [\widetilde{B}_{p,q}^s(S_D)]^5 \times [\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})]^5 \times [\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})]^4,\\
\mathbf{Y}_{p,q}^s &:= [B_{p,q}^{s+1}(S_D)]^5 \times [B_{p,q}^{s+1}(\Gamma_T^{(m)})]^5 \times [\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})]^4.
\end{aligned}$$

For $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \le q \le \infty$ we have the following mapping properties

$$\mathcal{N}_\tau^{(A)} : \mathbf{X}_p^s \to \mathbf{Y}_p^s \qquad \Big[\mathbf{X}_{p,q}^s \to \mathbf{Y}_{p,q}^s\Big].$$

Evidently, we can rewrite the system (3.9)–(3.11) as

$$\mathcal{N}_\tau^{(A)}\,\Phi = Y,$$

where $\Phi \in \mathbf{X}_p^s$ and $Y \in \mathbf{Y}_p^s$ are the above-introduced unknown and given vectors, respectively.

Theorem 3.4. *Let the conditions*

$$1 < p < \infty, \quad 1 \le q \le \infty, \quad \frac{1}{p} - 1 + \gamma'' < s + \frac{1}{2} < \frac{1}{p} + \gamma' \tag{3.13}$$

be satisfied with γ' and γ'' given by (3.18), (3.19), *and* (3.21). *Then the operators*

$$\mathcal{N}_\tau^{(A)} : \mathbf{X}_p^s \to \mathbf{Y}_p^s \qquad \Big[\mathbf{X}_{p,q}^s \to \mathbf{Y}_{p,q}^s\Big] \tag{3.14}$$

are invertible.

Proof. We prove the theorem in several steps. First we show that the operators (3.14) are Fredholm with zero index and afterwards we establish that the corresponding null-spaces are trivial.

Step 1. First of all let us remark that the operators

$$\begin{aligned} r_{_{S_D}}\mathcal{A}_\tau &: [\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})]^5 \to [B_{p,q}^{s+1}(S_D)]^5, \\ r_{_{\Gamma_T^{(m)}}}\mathcal{A}_\tau &: [\widetilde{B}_{p,q}^s(S_D)]^5 \to [B_{p,q}^{s+1}(\Gamma_T^{(m)})]^5, \end{aligned} \tag{3.15}$$

are compact since S_D and $\Gamma_T^{(m)}$ are disjoint, $\overline{S_D} \cap \overline{\Gamma_T^{(m)}} = \varnothing$. Further, we establish that the operators

$$\begin{aligned} r_{_{S_D}}\mathcal{A}_\tau \quad &: \quad [\widetilde{H}_2^{-1/2}(S_D)]^5 \to [H_2^{1/2}(S_D)]^5\,, \\ r_{_{\Gamma_T^{(m)}}}[\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] \quad &: \quad [\widetilde{H}_2^{-1/2}(\Gamma_T^{(m)})]^5 \to [H_2^{1/2}(\Gamma_T^{(m)})]^5 \end{aligned} \tag{3.16}$$

are strongly elliptic Fredholm pseudodifferential operators of order -1 with index zero. We remark that the principal homogeneous symbol matrices of these operators are strongly elliptic.

Using Green's formula and the Korn's inequality, for an arbitrary solution vector $U \in [H_2^1(\Omega)]^5 \equiv [W_2^1(\Omega)]^5$ to the homogeneous equation $A(\partial, \tau)U = 0$ in Ω by standard arguments we get

$$\Re\,\langle [\mathcal{T}U]^+, [U]^+\rangle_{\partial\Omega} \ge c_1\,||\,U\,||^2_{[H_2^1(\Omega)]^5} - c_2\,||\,U\,||^2_{[H_2^0(\Omega)]^5}\,.$$

Substituting here $U = V_\tau(\mathcal{P}_\tau^{-1}\zeta)$ with $\zeta \in [H_2^{-1/2}(\partial\Omega)]^5$ we arrive at the inequality

$$\Re\,\langle \zeta,\, r_{_{S_D}}\mathcal{H}_\tau\mathcal{P}_\tau^{-1}\zeta\rangle_{\partial\Omega} \ge c_1''\,||\,\zeta\,||^2_{[\widetilde{H}_2^{-1/2}(S_D)]^5} - c_2''\,||\,\zeta\,||^2_{[\widetilde{H}_2^{-3/2}(S_D)]^5}\,. \tag{3.17}$$

From (3.17) it follows that the operator

$$r_{_{S_D}}\mathcal{A}_\tau = r_{_{S_D}}\mathcal{H}_\tau\mathcal{P}_\tau^{-1} : [\widetilde{H}_2^{-1/2}(S_D)]^5 \to [H_2^{1/2}(S_D)]^5$$

is a strongly elliptic pseudodifferential Fredholm operator with index zero. Then the same is true for the operator (3.16) since the principal homogeneous symbol matrix of the operator $\mathcal{B}_\tau^{(m)}$ is nonnegative.

Therefore, due to the compactness of the operators (3.15), the operator (3.14) is Fredholm with index zero for $s = -1/2$, $p = 2$ and $q = 2$.

Step 2. With the help of the uniqueness Theorem 2.1 via representation formulas (3.2) and (3.3) with $G_0^{(m)} = 0$ and $G_0 = 0$ we can easily show that the operator (3.14) is injective for $s = -1/2$, $p = 2$ and $q = 2$. Since its index is zero, we conclude that it is surjective. Thus the operator (3.14) is invertible for $s = -1/2$, $p = 2$ and $q = 2$.

Step 3. To complete the proof for the general case we proceed as follows. We see that the following blockwise upper triangular operator

$$\mathcal{N}_\tau^{(A,0)} := \begin{bmatrix} r_{_{S_D}} \mathcal{A}_\tau & r_{_{S_D}} [\,0\,]_{5\times 5} & r_{_{S_D}} [\,0\,]_{5\times 4} \\ r_{_{\Gamma_T^{(m)}}} [\,0\,]_{5\times 5} & r_{_{\Gamma_T^{(m)}}} [\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] & r_{_{\Gamma_T^{(m)}}} [\,0\,]_{5\times 4} \\ r_{_{\Gamma_T^{(m)}}} [\,0\,]_{4\times 5} & r_{_{\Gamma_T^{(m)}}} I_{4\times 5} & r_{_{\Gamma_T^{(m)}}} I_4 \end{bmatrix}_{14\times 14}$$

is a compact perturbation of the operator (3.12). Therefore we have to investigate Fredholm properties of the operators

$$\begin{aligned} r_{_{S_D}} \mathcal{A}_\tau \quad &: \quad [\widetilde{B}^s_{p,q}(S_D]^5 \to [B^{s+1}_{p,q}(S_D)]^5, \\ r_{_{\Gamma_T^{(m)}}} [\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] \quad &: \quad [\widetilde{B}^s_{p,q}(\Gamma_T^{(m)})]^5 \to [B^{s+1}_{p,q}(\Gamma_T^{(m)})]^5. \end{aligned}$$

Let $\sigma_1(x, \xi_1, \xi_2) := \sigma(\mathcal{A}_\tau)(x, \xi_1, \xi_2)$ be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau$ and $\lambda_j^{(1)}(x)$, $j = \overline{1,5}$, be the eigenvalues of the matrix $\mathcal{D}_1(x) := [\sigma_1(x, 0, +1)]^{-1}\, \sigma_1(x, 0, -1)$ for $x \in \partial S_D$.

Similarly, let $\sigma_2(x, \xi_1, \xi_2) = \sigma(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})(x, \xi_1, \xi_2)$ be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}$ and $\lambda_j^{(2)}(x)$, $j = \overline{1,5}$, be the eigenvalues of the corresponding matrix $\mathcal{D}_2(x) := [\sigma_2(x, 0, +1)]^{-1}\, \sigma_2(x, 0, -1)$ for $x \in \partial\Gamma_T^{(m)}$. Recall that the curve $\partial\Gamma_T^{(m)}$ is the union of the curves where the interface intersects the exterior boundary, $\partial\Gamma^{(m)}$, and the crack edge, $\partial\Gamma_C^{(m)}$.

Further, we set

$$\gamma_1' := \inf_{x\in\partial S_D,\, 1\le j\le 5} \frac{1}{2\pi} \arg \lambda_j^{(1)}(x),$$

$$\gamma_1'' := \sup_{x\in\partial S_D,\, 1\le j\le 5} \frac{1}{2\pi} \arg \lambda_j^{(1)}(x), \tag{3.18}$$

$$\gamma_2' := \inf_{x\in\partial \Gamma_T^{(m)},\, 1\le j\le 5} \frac{1}{2\pi} \arg \lambda_j^{(2)}(x),$$

$$\gamma_2'' := \sup_{x\in\partial \Gamma_T^{(m)},\, 1\le j\le 5} \frac{1}{2\pi} \arg \lambda_j^{(2)}(x). \tag{3.19}$$

In can be shown that one of the eigenvalues is real and equals to 1. Therefore

$$\gamma_1' \leq 0, \qquad \gamma_1'' \geq 0. \tag{3.20}$$

Note that γ_j' and γ_j'' $(j = 1, 2)$ depend on the material parameters, in general, and belong to the interval $(-\frac{1}{2}, \frac{1}{2})$. We put

$$\gamma' := \min\{\gamma_1', \gamma_2'\}, \qquad \gamma'' := \max\{\gamma_1'', \gamma_2''\}. \tag{3.21}$$

In view of (3.20) we have

$$-\frac{1}{2} < \gamma' \leq 0 \leq \gamma'' < \frac{1}{2}\,. \tag{3.22}$$

Due to the general theory of pseudodifferential equations on manifolds with boundary (see, e.g., Theorem 8.1 in [7], [10]) we conclude that if the parameters $r_1, r_2 \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, satisfy the conditions

$$\frac{1}{p} - 1 + \gamma_1'' < r_1 + \frac{1}{2} < \frac{1}{p} + \gamma_1', \qquad \frac{1}{p} - 1 + \gamma_2'' < r_2 + \frac{1}{2} < \frac{1}{p} + \gamma_2',$$

then the operators

$$\begin{aligned} r_{S_D}\mathcal{A}_\tau \quad &: \quad [\widetilde{B}^{r_1}_{p,q}(S_D)]^5 \to [B^{r_1+1}_{p,q}(S_D)]^5\,, \\ r_{\Gamma_T^{(m)}}[\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] \quad &: \quad [\widetilde{B}^{r_2}_{p,q}(\Gamma_T^{(m)})]^5 \to [B^{r_2+1}_{p,q}(\Gamma_T^{(m)})]^5\,, \end{aligned}$$

are Fredholm operators with index zero.

Therefore, if the conditions (3.13) are satisfied, then the above operators are Fredholm with zero index. Consequently, the operators (3.14) are Fredholm with zero index and are invertible due to the results obtained in Step 2. □

Now we can formulate the basic existence and uniqueness results.

Theorem 3.5. *Let the inclusions* (2.15) *and compatibility conditions* (3.8) *hold and let*

$$\frac{4}{3 - 2\gamma''} < p < \frac{4}{1 - 2\gamma'}\,. \tag{3.23}$$

Then the interface crack problem (*ICP-A*) *has a unique solution which can be represented by formulas* (3.2) *and* (3.3), *where the densities* ψ, h, *and* $h^{(m)}$ *are to be determined from the system* (3.4)–(3.7) (*i.e., from the system* (3.9)–(3.11)).

Moreover, the vector functions $G_0 + \psi + h$ *and* $G_0^{(m)} + h^{(m)}$ *are defined uniquely by the above systems and do not depend on the extension operators.*

Proof. For p satisfying (3.23) and $s = -\frac{1}{p}$ it immediately follows that the pair $(U^{(m)}, U) \in [W^1_p(\Omega^{(m)})]^4 \times [W^1_p(\Omega)]^5$ given by (3.2)–(3.3) represents a solution to the interface crack problem (ICP-A). Now we show the uniqueness of solutions.

Due to the inequalities (3.22)

$$p = 2 \in \Big(\frac{4}{3 - 2\gamma''}\,,\ \frac{4}{1 - 2\gamma'}\Big).$$

Therefore the unique solvability for $p = 2$ is a consequence of Theorem 2.1.

To show the uniqueness result for all other values of p from the interval (3.23) we proceed as follows. Let a pair

$$(U^{(m)}, U) \in [W_p^1(\Omega^{(m)})]^4 \times [W_p^1(\Omega)]^5$$

with p satisfying (3.23) be a solution to the homogeneous interface crack problem (ICP-A). Then, it is evident that

$$\begin{aligned} &\{ U^{(m)} \}^+ \in [B_{p,p}^{1-\frac{1}{p}}(\partial\Omega^{(m)})]^4, \qquad &\{ U \}^+ \in [B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)]^5, \\ &\{ \mathcal{T}^{(m)} U^{(m)} \}^+ \in [B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)})]^4, \quad &\{ \mathcal{T} U \}^+ \in [B_{p,p}^{-\frac{1}{p}}(\partial\Omega)]^5, \end{aligned}$$

and the vectors $U^{(m)}$ and U in $\Omega^{(m)}$ and Ω respectively are representable as

$$\begin{aligned} &U^{(m)} = V_\tau^{(m)}\big(\big[\mathcal{P}_\tau^{(m)} \big]^{-1} h^{(m)} \big) \ \text{ in } \ \Omega^{(m)} \ \text{ with } \ h^{(m)} = \{ \mathcal{T}^{(m)} U^{(m)} \}^+, \\ &U = V_\tau\big(\mathcal{P}_\tau^{-1} \chi \big) \ \text{ in } \ \Omega \ \text{ with } \ \chi = \{\mathcal{T}U\}^+ + \beta\, \{U\}^+. \end{aligned}$$

Moreover, due to the homogeneous boundary, transmission and crack conditions, and since $\operatorname{supp}\beta \subset S_D$, we have

$$h^{(m)} \in [\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)})]^4, \ \ \chi = h + \psi, \ \ \psi \in \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D) \big]^5, \ \ h \in \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)}) \big]^5.$$

By the same arguments as above we arrive at the homogeneous system

$$\mathcal{N}_\tau^{(A)}\, \Phi = 0 \ \text{ with } \ \Phi := (\psi, \ h, \ h^{(m)})^\top \in \mathbf{X}_p^{-\frac{1}{p}}.$$

Due to Theorem 3.4, it follows that $\Phi = 0$ and we conclude $U^{(m)} = 0$ in $\Omega^{(m)}$ and $U = 0$ in Ω.

The last assertion of the theorem is trivial and is a ready consequence of the fact that if the single layer potentials (3.2) and (3.3) vanish identically in $\Omega^{(m)}$ and Ω, respectively, then the corresponding densities vanish as well. □

Finally, we formulate the following regularity result.

Theorem 3.6. *Let $\alpha > 0$ be noninteger and*

$$\begin{aligned} &Q_k \in B_{\infty,\infty}^{\alpha-1}(S_N), \ f_k \in C^\alpha(\overline{S_D}), \ f_k^{(m)} \in C^\alpha(\overline{\Gamma_T^{(m)}}), \ \widetilde{Q}_k \in B_{\infty,\infty}^{\alpha-1}(\Gamma_C^{(m)}), \\ &F_j^{(m)} \in B_{\infty,\infty}^{\alpha-1}(\Gamma_T^{(m)}), \ \widetilde{Q}_j^{(m)} \in B_{\infty,\infty}^{\alpha-1}(\Gamma_C^{(m)}), \ Q_j^{(m)} \in B_{\infty,\infty}^{\alpha-1}(S_N^{(m)}), \end{aligned}$$

where $k = \overline{1,5}$, $j = \overline{1,4}$, and the compatibility conditions

$$\widetilde{F}_j^{(m)} := F_j^{(m)} - r_{_{\Gamma_T^{(m)}}} G_{0j} - r_{_{\Gamma_T^{(m)}}} G_{0j}^{(m)} \in r_{_{\Gamma_T^{(m)}}} \widetilde{B}_{\infty,\infty}^{\alpha-1}(\Gamma_T^{(m)}), \ j = \overline{1,4},$$

be satisfied. Then the solution vectors have the following regularity property

$$U^{(m)} \in \bigcap_{\alpha' < \kappa} [C^{\alpha'}(\overline{\Omega^{(m)}})]^4, \qquad U \in \bigcap_{\alpha' < \kappa} [C^{\alpha'}(\overline{\Omega})]^5,$$

where $\kappa = \min\{\alpha, \ \gamma' + \frac{1}{2}\} > 0$.

A more detailed analysis based on the asymptotic expansions of solutions (see [2]) shows that for sufficiently smooth boundary data (e.g., C^∞-smooth data say) the principal dominant singular terms of the solution vectors $U^{(m)}$ and U near the exceptional curves ∂S_D and $\partial\Gamma_T^{(m)}$ can be represented as a product of a "good" vector function and a singular factor of the form $[\ln\varrho(x)]^{m_k-1}[\varrho(x)]^{\alpha_k+i\beta_k}$, where $\varrho(x)$ is the distance from a reference point x to the exceptional curves. Therefore, near these curves the dominant singular terms of the corresponding generalized stress vectors $\mathcal{T}^{(m)}U^{(m)}$ and $\mathcal{T}U$ are represented as a product of a "good" vector function and the factor $[\ln\varrho(x)]^{m_k-1}[\varrho(x)]^{-1+\alpha_k+i\beta_k}$. The exponents $\alpha_k+i\beta_k$ are related to the eigenvalues $\lambda_k^{(1)}(x)$ and $\lambda_k^{(2)}(x)$ of the above-introduced matrices $\mathcal{D}_1(x)$ and $\mathcal{D}_2(x)$ by the equalities

$$\alpha_k=\frac{1}{2}+\frac{\arg\lambda_k}{2\pi}, \quad \beta_k=-\frac{\ln|\lambda_k|}{2\pi}, \quad k=\overline{1,5}.$$

Here $\lambda_k\in\{\lambda_1^{(1)}(x),\dots,\lambda_5^{(1)}(x)\}$ for $x\in\partial S_D$, and $\lambda_k\in\{\lambda_1^{(2)}(x),\dots,\lambda_5^{(2)}(x)\}$ for $x\in\partial\Gamma_T^{(m)}$. The numbers β_k are different from zero, in general, and describe the oscillating character of the stress singularities. In the above expressions the parameter m_k denotes the multiplicity of the eigenvalue λ_k.

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On Positive Solutions of p-Laplacian-type Equations

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Dedicated to Vladimir Maz'ya on the occasion of his 70th birthday

Abstract. Let Ω be a domain in $\mathbb{R}^d$, $d \geq 2$, and $1 < p < \infty$. Fix $V \in L^\infty_{\rm loc}(\Omega)$. Consider the functional Q and its Gâteaux derivative Q' given by

$$Q(u) := \frac{1}{p}\int_\Omega (|\nabla u|^p + V|u|^p)\mathrm{d}x, \quad Q'(u) := -\nabla\cdot(|\nabla u|^{p-2}\nabla u) + V|u|^{p-2}u.$$

In this paper we discuss a few aspects of relations between functional-analytic properties of the functional Q and properties of positive solutions of the equation $Q'(u) = 0$.

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1. Introduction and preliminaries

Properties of positive solutions of quasilinear elliptic equations, and in particular of equations with the p-Laplacian term in the principal part, have been extensively studied over the recent decades, (see, for example, [3, 4, 28, 32] and the references therein). Fix $p \in (1,\infty)$, a domain $\Omega \subseteq \mathbb{R}^d$ and a real-valued potential $V \in L^\infty_{\rm loc}(\Omega)$. The p-Laplacian equation in Ω with potential V is the equation of the form

$$-\Delta_p(u) + V|u|^{p-2}u = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\Delta_p(u) := \nabla\cdot(|\nabla u|^{p-2}\nabla u)$ is the celebrated p-Laplacian. This equation, in the semistrong sense, is a critical point equation for the functional

$$Q(u) = Q_V(u) := \frac{1}{p}\int_\Omega (|\nabla u|^p + V|u|^p)\ \mathrm{d}x \qquad u \in C_0^\infty(\Omega). \tag{1.2}$$

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So, we consider solutions of (1.1) in the following weak sense.

Definition 1.1. A function $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a (*weak*) *solution* of the equation

$$Q'(u) := -\Delta_p(u) + V|u|^{p-2}u = 0 \quad \text{in } \Omega, \tag{1.3}$$

if for every $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega (|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi + V|v|^{p-2}v\varphi)\,\mathrm{d}x = 0. \tag{1.4}$$

We say that a real function $v \in C^1_{\mathrm{loc}}(\Omega)$ is a *supersolution* (resp. *subsolution*) of the equation (1.3) if for every nonnegative $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega (|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi + V|v|^{p-2}v\varphi)\,\mathrm{d}x \geq 0 \text{ (resp. } \leq 0). \tag{1.5}$$

Next, we mention *local* properties of solutions of (1.3) that hold in any smooth subdomain $\Omega' \Subset \Omega$ (i.e., $\overline{\Omega'}$ is compact in Ω).

1. Smoothness and Harnack inequality. Weak solutions of (1.3) admit Hölder continuous first derivatives, and nonnegative solutions of (1.3) satisfy the Harnack inequality (see for example [14, 32, 34, 35, 38]).

2. Principal eigenvalue and eigenfunction. For any smooth subdomain $\Omega' \Subset \Omega$ consider the variational problem

$$\lambda_{1,p}(\Omega') := \inf_{u\in W_0^{1,p}(\Omega')} \frac{\int_{\Omega'}(|\nabla u|^p + V|u|^p)\,\mathrm{d}x}{\int_{\Omega'}|u|^p\,\mathrm{d}x}. \tag{1.6}$$

It is well known that for such a subdomain, (1.6) admits (up to a multiplicative constant) a unique minimizer φ [9, 13]. Moreover, φ is a positive solution of the quasilinear eigenvalue problem

$$\begin{cases} Q'(\varphi) = \lambda_{1,p}(\Omega')|\varphi|^{p-2}\varphi & \text{in } \Omega', \\ \varphi = 0 & \text{on } \partial\Omega'. \end{cases} \tag{1.7}$$

$\lambda_{1,p}(\Omega')$ and φ are called, respectively, the *principal eigenvalue and eigenfunction* of the operator Q' in Ω'.

3. Weak and strong maximum principles

Theorem 1.2 ([13] (see also [3, 4])). *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded $C^{1+\alpha}$-domain, where $0 < \alpha \leq 1$. Consider a functional Q of the form (1.2) with $V \in L^\infty(\Omega)$. The following assertions are equivalent:*

(i) *Q' satisfies the maximum principle: If u is a solution of the equation $Q'(u) = f \geq 0$ in Ω with some $f \in L^\infty(\Omega)$, and satisfies $u \geq 0$ on $\partial\Omega$, then u is nonnegative in Ω.*

(ii) *Q' satisfies the strong maximum principle: If u is a solution of the equation $Q'(u) = f \gneqq 0$ in Ω with some $f \in L^\infty(\Omega)$, and satisfies $u \geq 0$ on $\partial\Omega$, then $u > 0$ in Ω.*

(iii) $\lambda_{1,p}(\Omega) > 0$.

(iv) *For some* $0 \lneqq f \in L^\infty(\Omega)$ *there exists a positive strict supersolution* v *satisfying* $Q'(v) = f$ *in* Ω, *and* $v = 0$ *on* $\partial\Omega$.

(iv′) *There exists a positive strict supersolution* $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *satisfying* $Q'(v) = f \gneqq 0$ *in* Ω, *such that* $v|_{\partial\Omega} \in C^{1+\alpha}(\partial\Omega)$ *and* $f \in L^\infty(\Omega)$.

(v) *For each nonnegative* $f \in C^\alpha(\Omega) \cap L^\infty(\Omega)$ *there exists a unique weak nonnegative solution of the problem* $Q'(u) = f$ *in* Ω, *and* $u = 0$ *on* $\partial\Omega$.

4. Weak comparison principle. We recall also the following *weak comparison principle* (or WCP for brevity).

Theorem 1.3 ([13]). *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain of class* $C^{1,\alpha}$, *where* $0 < \alpha \le 1$, *and suppose that* $V \in L^\infty(\Omega)$. *Assume that* $\lambda_{1,p}(\Omega) > 0$ *and let* $u_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *satisfying* $Q'(u_i) \in L^\infty(\Omega)$, $u_i|_{\partial\Omega} \in C^{1+\alpha}(\partial\Omega)$, *where* $i = 1, 2$. *Suppose further that the following inequalities are satisfied*

$$\begin{cases} Q'(u_1) \le Q'(u_2) & in\ \Omega, \\ Q'(u_2) \ge 0 & in\ \Omega, \\ u_1 \le u_2 & on\ \partial\Omega, \\ u_2 \ge 0 & on\ \partial\Omega. \end{cases} \tag{1.8}$$

Then

$$u_1 \le u_2 \qquad in\ \Omega.$$

5. Strong comparison principle

Definition 1.4. We say that the *strong comparison principle* (or SCP for brevity) holds true for the functional Q if the conditions of Theorem 1.3 and possibly some additional conditions imply that $u_1 < u_2$ in Ω unless $u_1 = u_2$ in Ω.

Remark 1.5. It is well known that the SCP holds true for $p = 2$ in any dimension, and for p-harmonic functions in the plane. For sufficient conditions for the validity of the SCP see [2, 5, 7, 16, 32, 37] and the references therein. In [5] M. Cuesta and P. Takáč present a counterexample where the WCP holds true but the SCP does not.

Throughout this paper we assume that

$$Q(u) \ge 0 \qquad \forall u \in C_0^\infty(\Omega). \tag{1.9}$$

The following Allegretto-Piepenbrink-type theorem, links the existence of positive solutions with the positivity of Q.

Theorem 1.6 ([28, Theorem 2.3]). *Consider a functional* Q *of the form* (1.2). *The following assertions are equivalent:*

(i) *The functional* Q *is nonnegative on* $C_0^\infty(\Omega)$.

(ii) *Equation* (1.3) *admits a global positive solution.*

(iii) *Equation* (1.3) *admits a global positive supersolution.*

In this paper we survey further connections between functional-analytic properties of the functional Q and properties of its positive solutions. In particular, we review the following topics:

- A representation of the nonnegative functional Q as an integral of a nonnegative Lagrangian density and the existence of a useful equivalent nonnegative Lagrangian density with a simplified form (Section 2).
- The equivalence of several weak coercivity properties of Q. The characterization of the non-coercive case in terms of a Poincaré-type inequality, in terms of the existence of a generalized ground state, and in terms of the variational capacity of balls (Section 3).
- The identification of ground state as a global minimal solution (Section 4).
- A theorem of Liouville type connecting the behavior of a ground state of one functional with the existence of a ground state of another functional with a given 'decaying' subsolution (Section 5).
- A variational principle that characterizes solutions of minimal growth at infinity (Section 6).
- The existence of solutions to the inhomogeneous equation $Q'(u) = f$ in the absence of the ground state (Section 7).
- The dependence of weak coercivity on the potential and the domain (Section 8). In particular, in Theorem 8.6, we extend the result for $p = 2$ proved in [30, Theorem 2.9].
- Properties verified only in the linear case ($p = 2$), in particular, the definition of a natural functional space associated with the functional Q (Section 9).

2. Positive Lagrangian representations

Let $v \in C^1_{\mathrm{loc}}(\Omega)$ be a positive solution (resp. subsolution) of (1.3). Using the *positive Lagrangian representation* from [3, 4, 6], we infer that for every $u \in C_0^\infty(\Omega)$, $u \geq 0$,

$$Q(u) = \int_\Omega L(u, v)\,\mathrm{d}x, \qquad \text{resp. } Q(u) \leq \int_\Omega L(u, v)\,\mathrm{d}x, \tag{2.1}$$

where

$$L(u, v) := \frac{1}{p}\left[|\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u \cdot |\nabla v|^{p-2}\nabla v\right]. \tag{2.2}$$

It can be easily verified that $L(u, v) \geq 0$ in Ω [3, 4].

Let now $w := u/v$, where v is a positive solution of (1.3) and $u \in C_0^\infty(\Omega)$, $u \geq 0$. Then (2.1) implies that

$$Q(vw) = \frac{1}{p}\int_\Omega \left[|w\nabla v + v\nabla w|^p - w^p|\nabla v|^p - pw^{p-1}v|\nabla v|^{p-2}\nabla v \cdot \nabla w\right]\,\mathrm{d}x. \tag{2.3}$$

Similarly, if v is a nonnegative subsolution of (1.3), then

$$Q(vw) \leq \frac{1}{p}\int_\Omega \left[|w\nabla v + v\nabla w|^p - w^p|\nabla v|^p - pw^{p-1}v|\nabla v|^{p-2}\nabla v \cdot \nabla w\right]\,\mathrm{d}x. \tag{2.4}$$

Therefore, a nonnegative functional Q can be represented as the integral of a nonnegative Lagrangian L. In spite of the nonnegativity of the integrands in (2.1) and (2.3), the expression (2.2) of L contains an indefinite term which poses obvious difficulties for extending the domain of the functional to more general weakly differentiable functions. The next proposition shows that Q admits a two-sided estimate by a simplified Lagrangian containing only nonnegative terms. We call the functional associated with this simplified Lagrangian the *simplified energy.*

Let f and g be two nonnegative functions. We denote $f \asymp g$ if there exists a positive constant C such that $C^{-1}g \leq f \leq Cg$.

Proposition 2.1 ([26, Lemma 2.2])**.** *Let $v \in C^1_{\mathrm{loc}}(\Omega)$ be a positive solution of* (1.3)*. Then*

$$Q(vw) \asymp \int_\Omega v^2 |\nabla w|^2 \left(w|\nabla v| + v|\nabla w|\right)^{p-2} \,\mathrm{d}x \quad \forall w \in C^1_0(\Omega), w \geq 0. \tag{2.5}$$

In particular, for $p \geq 2$, we have

$$Q(vw) \asymp \int_\Omega \left(v^p |\nabla w|^p + v^2 |\nabla v|^{p-2} w^{p-2} |\nabla w|^2\right) \mathrm{d}x \quad \forall w \in C^1_0(\Omega), w \geq 0. \tag{2.6}$$

If v is only a nonnegative subsolution of (1.3)*, then for $1 < p < \infty$ we have*

$$Q(vw) \leq C \int_{\Omega \cap \{v>0\}} v^2 |\nabla w|^2 \left(w|\nabla v| + v|\nabla w|\right)^{p-2} \mathrm{d}x \quad \forall w \in C^1_0(\Omega), w \geq 0.$$

In particular, for $p \geq 2$ we have

$$Q(vw) \leq C \int_\Omega \left(v^p |\nabla w|^p + v^2 |\nabla v|^{p-2} w^{p-2} |\nabla w|^2\right) \mathrm{d}x \quad \forall w \in C^1_0(\Omega), w \geq 0.$$

Remark 2.2. It is shown in [26] that for $p > 2$ neither of the terms in the simplified energy (2.6) is dominated by the other, so that (2.6) cannot be further simplified.

3. Coercivity and ground state

It is well known (see [21]) that for a nonnegative Schrödinger operator P we have the following dichotomy: either there exists a strictly positive potential W such that the Schrödinger operator $P - W$ is nonnegative, or P admits a unique (generalized) ground state. It turns out that this statement is also true for nonnegative functionals of the form (1.2).

Definition 3.1. Let Q be a nonnegative functional on $C^\infty_0(\Omega)$ of the form (1.2). We say that a sequence $\{u_k\} \subset C^\infty_0(\Omega)$ of nonnegative functions is a *null sequence* of the functional Q in Ω, if there exists an open set $B \Subset \Omega$ such that $\int_B |u_k|^p \,\mathrm{d}x = 1$, and

$$\lim_{k\to\infty} Q(u_k) = \lim_{k\to\infty} \int_\Omega \left(|\nabla u_k|^p + V|u_k|^p\right) \mathrm{d}x = 0. \tag{3.1}$$

We say that a positive function $v \in C^1_{\mathrm{loc}}(\Omega)$ is a *ground state* of the functional Q in Ω if v is an $L^p_{\mathrm{loc}}(\Omega)$ limit of a null sequence of Q. If $Q \geq 0$, and Q admits a ground state in Ω, we say that Q is *critical* in Ω.

Remark 3.2. The requirement that $\{u_k\} \subset C_0^\infty(\Omega)$ can clearly be weakened by assuming only that $\{u_k\} \subset W_0^{1,p}(\Omega)$. Also, as it follows from Theorem 3.4, the requirement $\int_B |u_k|^p \, \mathrm{d}x = 1$ can be replaced by $\int_B |u_k|^p \, \mathrm{d}x \asymp 1$ or by $\int_B u_k \, \mathrm{d}x \asymp 1$.

The following statements are based on rephrased statements of [28, Theorem 1.6], [29, Theorem 4.3] and [36, Proposition 3.1] (cf. [21, 27] for the case $p = 2$).

Theorem 3.3. *Suppose that the functional Q is nonnegative on $C_0^\infty(\Omega)$.*

1. *Any ground state v is a positive solution of* (1.3).
2. *Q admits a ground state v if and only if* (1.3) *admits a unique positive supersolution.*
3. *Q is critical in Ω if and only if Q admits a null sequence that converges locally uniformly in Ω.*
4. *If Q admits a ground state v, then the following Poincaré type inequality holds: There exists a positive continuous function W in Ω, such that for every $\psi \in C_0^\infty(\Omega)$ satisfying $\int \psi v \, \mathrm{d}x \neq 0$ there exists a constant $C > 0$ such that:*

$$Q(u) + C \left| \int_\Omega \psi u \, \mathrm{d}x \right|^p \geq C^{-1} \int_\Omega W \left(|\nabla u|^p + |u|^p \right) \mathrm{d}x \qquad \forall u \in C_0^\infty(\Omega). \tag{3.2}$$

The following theorem slightly extends [28, Theorem 1.6] in the spirit of [36, Proposition 3.1].

Theorem 3.4. *Suppose that the functional Q is nonnegative on $C_0^\infty(\Omega)$. The following statements are equivalent.*

(a) *Q does not admit a ground state in Ω.*

(b) *There exists a continuous function $W > 0$ in Ω such that*

$$Q(u) \geq \int_\Omega W(x) |u(x)|^p \, \mathrm{d}x \qquad \forall u \in C_0^\infty(\Omega). \tag{3.3}$$

(c) *There exists a continuous function $W > 0$ in Ω such that*

$$Q(u) \geq \int_\Omega W(x) \left(|\nabla u(x)|^p + |u(x)|^p \right) \mathrm{d}x \qquad \forall u \in C_0^\infty(\Omega). \tag{3.4}$$

(d) *There exists an open set $B \Subset \Omega$ and $C_B > 0$ such that*

$$Q(u) \geq C_B \left| \int_B u(x) \, \mathrm{d}x \right|^p \qquad \forall u \in C_0^\infty(\Omega). \tag{3.5}$$

Suppose further that $d > p$. Then Q does not admit a ground state in Ω if and only if there exists a continuous function $W > 0$ in Ω such that

$$Q(u) \geq \left(\int_\Omega W(x) |u(x)|^{p^*} \, \mathrm{d}x \right)^{p/p^*} \qquad \forall u \in C_0^\infty(\Omega), \tag{3.6}$$

where $p^* = pd/(d - p)$ is the critical Sobolev exponent.

Definition 3.5. A nonnegative functional Q on $C_0^\infty(\Omega)$ of the form (1.2) which is not critical is said to be *subcritical* (or *weakly coercive*) in Ω.

Example 3.6. Consider the functional $Q(u) := \int_{\mathbb{R}^d} |\nabla u|^p \,\mathrm{d}x$. It follows from [20, Theorem 2] that if $d \le p$, then Q admits a ground state $\varphi =$ constant in $\mathbb{R}^d$. On the other hand, if $d > p$, then

$$u(x) := \left[1 + |x|^{p/(p-1)}\right]^{(p-d)/p}, \qquad v(x) := \text{constant}$$

are two positive supersolutions of the equation $-\Delta_p u = 0$ in $\mathbb{R}^d$. Therefore, Q is weakly coercive in $\mathbb{R}^d$.

Example 3.7. Let $d > 1$, $d \ne p$, and $\Omega := \mathbb{R}^d \setminus \{0\}$ be the punctured space. The following celebrated Hardy's inequality holds true:

$$Q_\lambda(u) := \int_\Omega \left(|\nabla u|^p - \lambda \frac{|u|^p}{|x|^p} \right) \mathrm{d}x \ge 0 \qquad u \in C_0^\infty(\Omega), \tag{3.7}$$

whenever $\lambda \le c^*_{p,d} := |(p-d)/p|^p$. Clearly, if $\lambda < c^*_{p,d}$, then $Q_\lambda(u)$ is weakly coercive. On the other hand, the proof of Theorem 1.3 in [31] shows that Q_λ with $\lambda = c^*_{p,d}$ admits a null sequence. It can be easily checked that the function $v(r) := |r|^{(p-d)/p}$ is a positive solution of the corresponding radial equation:

$$-|v'|^{p-2} \left[(p-1)v'' + \frac{d-1}{r} v'\right] - c^*_{p,d} \frac{|v|^{p-2}v}{r^p} = 0 \qquad r \in (0, \infty).$$

Therefore, $\varphi(x) := |x|^{(p-d)/p}$ is the ground state of the equation

$$-\Delta_p u - c^*_{p,d} \frac{|u|^{p-2}u}{|x|^p} = 0 \qquad \text{in } \Omega. \tag{3.8}$$

Note that $\varphi \notin W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ for $p \ne d$. In particular, φ is not a positive supersolution of the equation $\Delta_p u = 0$ in $\mathbb{R}^d$.

In [39, 40] Troyanov has established a relationship between the p-capacity of closed balls in a Riemannian manifold $\mathcal{M}$ and the p-parabolicity of $\mathcal{M}$ with respect the p-Laplacian. We extend his definition and result to our case.

Definition 3.8. Suppose that the functional Q is nonnegative on $C_0^\infty(\Omega)$. Let $K \Subset \Omega$ be a compact set. The *Q-capacity* of K in Ω is defined by

$$\mathrm{Cap}_Q(K, \Omega) := \inf\{Q(u) \mid u \in C_0^\infty(\Omega),\ u \ge 1 \ \text{ on } K\}.$$

Corollary 3.9. *Suppose that the functional Q is nonnegative on $C_0^\infty(\Omega)$. Then Q is critical in Ω if and only if the Q-capacity of each closed ball in Ω is zero.*

4. Ground states and minimal growth at infinity

Definition 4.1. Let K_0 be a compact set in Ω. A positive solution u of the equation $Q'(u) = 0$ in $\Omega \setminus K_0$ is said to be a *positive solution of minimal growth in a neighborhood of infinity in* Ω (or $u \in \mathcal{M}_{\Omega,K_0}$ for brevity) if for any compact set K in Ω, with a smooth boundary, such that $K_0 \Subset \mathrm{int}(K)$, and any positive supersolution $v \in C((\Omega \setminus K) \cup \partial K)$ of the equation $Q'(u) = 0$ in $\Omega \setminus K$, the inequality $u \le v$ on ∂K implies that $u \le v$ in $\Omega \setminus K$.

A (global) positive solution u of the equation $Q'(u) = 0$ in Ω, which has minimal growth in a neighborhood of infinity in Ω (i.e., $u \in \mathcal{M}_{\Omega,\emptyset}$) is called a *global minimal solution of the equation* $Q'(u) = 0$ *in* Ω.

Theorem 4.2 ([29, Theorem 5.1], **cf.** [28]). *Suppose that* $1 < p < \infty$*, and* Q *is nonnegative on* $C_0^\infty(\Omega)$*. Then for any* $x_0 \in \Omega$ *the equation* $Q'(u) = 0$ *has a positive solution* $u \in \mathcal{M}_{\Omega,\{x_0\}}$.

We have the following connection between the existence of a global minimal solutions and weak coercivity.

Theorem 4.3 ([29, Theorem 5.2], **cf.** [28]). *Let* $1 < p < \infty$*, and assume that* Q *is nonnegative on* $C_0^\infty(\Omega)$*. Then* Q *is subcritical in* Ω *if and only if the equation* $Q'(u) = 0$ *in* Ω *does not admit a global minimal solution of the equation* $Q'(u) = 0$ *in* Ω*. In particular,* u *is ground state of the equation* $Q'(u) = 0$ *in* Ω *if and only* u *is a global minimal solution of this equation.*

Consider a positive solution u of the equation $Q'(u) = 0$ in a punctured neighborhood of x_0 which has a nonremovable singularity at $x_0 \in \mathbb{R}^d$. Without loss of generality we may assume that $x_0 = 0$. If $1 < p \le d$, then the behavior of u near an isolated singularity is well understood. Indeed, due to a result of L. Véron (see [28, Lemma 5.1]), we have that

$$u(x) \sim \begin{cases} |x|^{\alpha(d,p)} & p < d, \\ -\log|x| & p = d, \end{cases} \qquad \text{as } x \to 0, \tag{4.1}$$

where $\alpha(d,p) := (p-d)/(p-1)$, and $f \sim g$ means that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = C$$

for some positive constant C. In particular, $\lim_{x\to 0} u(x) = \infty$.

Assume now that $p > d$. A general question is whether in this case, any positive solution of the equation $Q'(u) = 0$ in a punctured ball centered at x_0 can be continuously extended at x_0 (see [17] for partial results).

Under the assumption that $u \asymp 1$ near the isolated point the answer is given by Lemma 5.3 in [29]:

Lemma 4.4. *Assume that* $p > d$*, and let* v *be a positive solution of the equation* $Q'(u) = 0$ *in a punctured neighborhood of* x_0 *satisfying* $u \asymp 1$ *near* x_0*. Then* u *can be continuously extended at* x_0.

The following statement combines the second part of [28, Theorem 5.4], where the case $1 < p \leq d$ is considered with [29, Theorem 5.3] which deals with the case $p > d$.

Theorem 4.5. *Let $x_0 \in \Omega$, and let $u \in \mathcal{M}_{\Omega,\{x_0\}}$. Then Q is subcritical in Ω if and only if u has a nonremovable singularity at x_0.*

5. Liouville theorems

In [26] we use some of the positivity properties of the nonnegative functional Q discussed in the previous sections to prove a Liouville comparison principle for equations $Q'(u) = 0$ in Ω (see Theorem 9.1 for the case $p = 2$).

Theorem 5.1. *Let Ω be a domain in $\mathbb{R}^d$, $d \geq 1$, and let $p \in (1, \infty)$. For $j = 0, 1$, let $V_j \in L^\infty_{\mathrm{loc}}(\Omega)$, and let*

$$Q_j(u) := \int_\Omega \left(|\nabla u(x)|^p + V_j(x)|u(x)|^p\right) \mathrm{d}x \qquad u \in C_0^\infty(\Omega).$$

Assume that the following assumptions hold true.

(i) *The functional Q_1 admits a ground state φ in Ω.*
(ii) *$Q_0 \geq 0$ on $C_0^\infty(\Omega)$, and the equation $Q_0'(u) = 0$ in Ω admits a subsolution $\psi \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfying $\psi_+ \neq 0$, where $\psi_+(x) := \max\{0, \psi(x)\}$.*
(iii) *The following inequality holds in Ω*

$$\psi_+ \leq C\varphi, \tag{5.1}$$

where $C > 0$ is a positive constant.

(iv) *The following inequality holds in Ω*

$$|\nabla \psi_+|^{p-2} \leq C|\nabla \varphi|^{p-2}, \tag{5.2}$$

where $C > 0$ is a positive constant.

Then the functional Q_0 admits a ground state in Ω, and ψ is the ground state. In particular, ψ is (up to a multiplicative constant) the unique positive supersolution of the equation $Q_0'(u) = 0$ in Ω.

Remark 5.2. Condition (5.2) is redundant for $p = 2$. For $p \neq 2$ it is equivalent to the assumption that the following inequality holds in Ω:

$$\begin{cases} |\nabla \psi_+| \leq C|\nabla \varphi| & \text{if } p > 2, \\ |\nabla \psi_+| \geq C|\nabla \varphi| & \text{if } p < 2, \end{cases} \tag{5.3}$$

where $C > 0$ is a positive constant.

Remark 5.3. This theorem holds if, in addition to (5.1), one assumes instead of $|\nabla \psi_+|^{p-2} \leq C|\nabla \varphi|^{p-2}$ in Ω (see (5.2)), that the following inequality holds true in Ω

$$\psi_+^2 |\nabla \psi_+|^{p-2} \leq C\varphi^2 |\nabla \varphi|^{p-2}, \tag{5.4}$$

where $C > 0$ is a positive constant.

Remark 5.4. Suppose that $1 < p < 2$, and assume that the ground state $\varphi > 0$ of the functional Q_1 is such that $w = \mathbf{1}$ is a ground state of the functional

$$E_1^{\varphi}(w) = \int_{\Omega} \varphi^p |\nabla w|^p \, \mathrm{d}x, \tag{5.5}$$

that is, there is a sequence $\{w_k\} \subset C_0^\infty(\Omega)$ of nonnegative functions satisfying $E_1^{\varphi}(w_k) \to 0$, and $\int_B |w_k|^p = 1$ for a fixed ball $B \Subset \Omega$ with volume 1 (this implies that $w_k \to \mathbf{1}$ in $L^p_{\mathrm{loc}}(\Omega)$). In this case, the conclusion of Theorem 5.1 holds if there is a nonnegative subsolution ψ_+ of $Q_0'(u) = 0$ satisfying (5.1) alone, without an assumption on the gradients (like (5.2) or (5.4)).

Remark 5.5. Condition (5.2) is essential when $p > 2$, and presumably also when $p < 2$. When $p > 2$, $\Omega = \mathbb{R}^d$, and V is radially symmetric, Proposition 4.2 in [26] shows that the simplified energy functional is not equivalent to either of its two terms that lead to conditions (5.1) and (5.2), respectively.

Example 5.6. Assume that $1 \le d \le p \le 2$, $p > 1$, $\Omega = \mathbb{R}^d$, and consider the functional $Q_1(u) := \int_{\mathbb{R}^d} |\nabla u|^p \, \mathrm{d}x$. By Example 3.6, the functional Q_1 admits a ground state $\varphi =$ constant in $\mathbb{R}^d$.

Let Q_0 be a functional of the form (1.2) satisfying $Q_0 \ge 0$ on $C_0^\infty(\mathbb{R}^d)$. Let $\psi \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $\psi_+ \ne 0$ be a subsolution of the equation $Q_0'(u) = 0$ in $\mathbb{R}^d$, such that $\psi_+ \in L^\infty(\mathbb{R}^d)$. It follows from Theorem 5.1 that ψ is the ground state of Q_0 in $\mathbb{R}^d$. In particular, ψ is (up to a multiplicative constant) the unique positive supersolution and unique bounded solution of the equation $Q_0'(u) = 0$ in $\mathbb{R}^d$. Note that there is no assumption on the behavior of the potential V_0 at infinity. This result generalizes some striking Liouville theorems for Schrödinger operators on $\mathbb{R}^d$ that hold for $d = 1, 2$ and $p = 2$ (see [24, Theorems 1.4–1.6]).

Example 5.7. Let $1 < p < \infty$, $d > 1$, $d \ne p$, and $\Omega := \mathbb{R}^d \setminus \{0\}$ be the punctured space. Let Q_0 be a functional of the form (1.2) satisfying $Q_0 \ge 0$ on $C_0^\infty(\Omega)$. Let $\psi \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $\psi_+ \ne 0$ be a subsolution of the equation $Q_0'(u) = 0$ in Ω, satisfying

$$\psi_+(x) \le C|x|^{(p-d)/p} \qquad x \in \Omega. \tag{5.6}$$

When $p > 2$, we require in addition that the following inequality is satisfied

$$\psi_+(x)^2 |\nabla \psi_+(x)|^{p-2} \le C|x|^{2-d} \qquad x \in \Omega. \tag{5.7}$$

It follows from Theorem 5.1, Remark 5.3, Remark 5.4 and Example 3.7 that ψ is the ground state of Q_0 in Ω. Let φ be the ground state of the Hardy functional. The reason that (5.7) is stated only for $p > 2$ hinges on the property that for $1 < p < 2$ the functional

$$E_1^{\varphi}(w) = \int_{\Omega} |x|^{p-d} |\nabla w|^p \, \mathrm{d}x \tag{5.8}$$

admits a ground state $\mathbf{1}$, so Remark 5.4 applies.

Next, we present a family of functionals Q_0 for which the conditions of Example 5.7 are satisfied.

Example 5.8. Let $d \geq 2$, $1 < p < d$, $\alpha \geq 0$, and $\Omega := \mathbb{R}^d \setminus \{0\}$. Let

$$W_\alpha(x) := -\left(\frac{d-p}{p}\right)^p \frac{\alpha dp/(d-p) + |x|^{\frac{p}{p-1}}}{\left(\alpha + |x|^{\frac{p}{p-1}}\right)^p}.$$

Note that if $\alpha = 0$ this is the Hardy potential as in Example 3.7. If Q_0 is the functional (1.2) with the potential $V_0 := W_\alpha$, then

$$\psi_\alpha(x) := \left(\alpha + |x|^{\frac{p}{p-1}}\right)^{-\frac{(d-p)(p-1)}{p^2}}$$

is a solution of $Q_0'(u) = 0$ in Ω, and therefore $Q_0 \geq 0$ on $C_0^\infty(\Omega)$. Moreover, one can use Example 5.7 to show that ψ_α is a ground state of Q_0. Note first that $\psi = \psi_\alpha$ satisfies (5.6). If $\frac{d}{d-1} < p < d$, then ψ_α satisfies also (5.7) and therefore, it is a ground state in this case. In the remaining case $p \leq \frac{d}{d-1} \leq 2$, Example 3.7 concludes that ψ_α is a ground state from the property of the functional (5.8).

6. Variational principle for solutions of minimal growth and comparison principle

The aim of this section is to represent positive solutions of minimal growth in a neighborhood of infinity in Ω as a limit of a modified null sequence.

Theorem 6.1 ([29, Theorem 7.1]). *Suppose that $1 < p < \infty$, and let Q_V be nonnegative on $C_0^\infty(\Omega)$. Let $\Omega_1 \Subset \Omega$ be an open set, and let $u \in C(\Omega \setminus \Omega_1)$ be a positive solution of the equation $Q_V'(u) = 0$ in $\Omega \setminus \overline{\Omega_1}$ satisfying $|\nabla u| \neq 0$ in $\Omega \setminus \overline{\Omega_1}$.*

Then $u \in \mathcal{M}_{\Omega,\overline{\Omega_1}}$ if for every smooth open set Ω_2 satisfying $\Omega_1 \Subset \Omega_2 \Subset \Omega$, and an open set $B \Subset (\Omega \setminus \overline{\Omega_2})$ there exists a sequence $\{u_k\} \subset C_0^\infty(\Omega)$, $u_k \geq 0$, such that for all $k \in \mathbb{N}$, $\int_B |u_k|^p \, \mathrm{d}x = 1$, and

$$\lim_{k\to\infty} \int_{\Omega\setminus\overline{\Omega}_2} L(u_k, u) \, \mathrm{d}x = 0, \tag{6.1}$$

where L is the Lagrangian given by (2.2).

Conjecture 6.2. We conjecture that for $p \neq 2$ a positive global solution of the equation $Q_V'(u) = 0$ in Ω satisfying $u \in \mathcal{M}_{\Omega,\overline{\Omega_1}}$ for some smooth open set $\Omega_1 \Subset \Omega$ is a global minimal solution.

Remark 6.3. The validity of Conjecture 6.2 seems to be related to the SCP. We note that if Conjecture 6.2 holds true, then the condition of Theorem 6.1 is also necessary (cf. Section 9.3).

Finally, we formulate a sub-supersolution comparison principle for our singular elliptic equation.

Theorem 6.4 (Comparison Principle [29]**).** *Assume that the functional Q_V is nonnegative on $C_0^\infty(\Omega)$. Fix smooth open sets $\Omega_1 \Subset \Omega_2 \Subset \Omega$. Let $u, v \in W^{1,p}_{\mathrm{loc}}(\Omega \setminus \Omega_1) \cap C(\Omega \setminus \Omega_1)$ be, respectively, a positive subsolution and a supersolution of the equation $Q'_V(w) = 0$ in $\Omega \setminus \overline{\Omega_1}$ such that $u \le v$ on $\partial\Omega_2$.*

Assume further that $Q'_V(u) \in L^\infty_{\mathrm{loc}}(\Omega \setminus \Omega_1)$, $|\nabla u| \neq 0$ in $\Omega \setminus \overline{\Omega_1}$, and that there exist an open set $B \Subset (\Omega \setminus \overline{\Omega_2})$ and a sequence $\{u_k\} \subset C_0^\infty(\Omega)$, $u_k \ge 0$, such that

$$\int_B |u_k|^p \,\mathrm{d}x = 1 \quad \forall k \ge 1, \text{ and } \quad \lim_{k\to\infty} \int_{\Omega\setminus\overline{\Omega}_1} L(u_k, u)\,\mathrm{d}x = 0. \tag{6.2}$$

Then $u \le v$ on $\Omega \setminus \Omega_2$.

7. Solvability of nonhomogeneous equation

In this section we discuss some results of [36] concerning the solvability of the nonhomogeneous equation

$$Q'_V(u) = f \qquad \text{in } \Omega, \tag{7.1}$$

where Q_V is the nonnegative functional (1.2). In some cases, e.g., $V \ge 0$ or $p = 2$, the nonnegativity of Q_V implies that Q_V is convex. In general, however, Q_V might be nonconvex. For $p > 2$, see the elementary one-dimensional example at the end of [8], and also the proof of [13, Theorem 7]. For $p < 2$, see Example 2 in [12].

If Q_V is convex and weakly coercive, then (7.1) can be easily solved by defining a Banach space X as a completion of $C_0^\infty(\Omega)$ with respect to the norm $Q_V(\cdot)^{1/p}$ (see the discussion of the analogous space for $p = 2$ in Section 9 below). Such space is continuously imbedded into $W^{1,p}_{\mathrm{loc}}(\Omega)$ by (3.4). It follows that for every $f \in X^*$ the functional

$$u \mapsto Q_V(u) - \langle f, u\rangle \qquad u \in X,$$

has a minimum that solves (7.1).

Note that the requirement of weak coercivity cannot be removed. Indeed, if $p = 2$, Ω is smooth and bounded, and $V = 0$, then the corresponding ground state φ is the first eigenfunction of the Dirichlet Laplacian with an eigenvalue λ_0, and there is no solution $u \in W_0^{1,2}(\Omega)$ to the equation

$$(-\Delta - \lambda_0)u = f \qquad \text{in } \Omega \tag{7.2}$$

unless $\int_\Omega \varphi(x) f(x)\,\mathrm{d}x = 0$.

In order to address the nonconvex case, we use the following setup from convex analysis (see [10, Chapt. I] for details.) The *polar* (or *conjugate*) functional to Q_V is defined by

$$Q_V^*(f) := \sup_{u\in C_0^\infty(\Omega)} [\langle u, f\rangle - Q_V(u)] \qquad f \in \mathcal{D}'(\Omega). \tag{7.3}$$

Notice that Q_V is positively homogeneous of degree p, and consequently, Q_V^* is positively homogeneous functional of degree p', where $p' = p/(p-1)$. The (*effective* or *natural*) *domain* X^* of Q_V^* is defined naturally by

$$X^* = \{f \in \mathcal{D}'(\Omega)\colon Q_V^*(f) < \infty\}. \tag{7.4}$$

The definition of $Q_V^*(f)$ in (7.3) yields immediately the well-known Fenchel-Young inequality

$$|\langle u, f\rangle| \le Q_V(u) + Q_V^*(f), \tag{7.5}$$

and equivalently, the Hölder inequality

$$|\langle u, f\rangle| \le (p\, Q_V(u))^{1/p} \, (p'\, Q_V^*(f))^{1/p'} . \tag{7.6}$$

One can easily verify that X^* is a linear subspace of $\mathcal{D}'(\Omega)$ and

$$\|f\|_* := (p' Q_V^*(f))^{1/p'} \tag{7.7}$$

defines a norm on X^*. In particular, $\|f\|_* = 0$ implies $f = 0$, as a consequence of (7.6) combined with the separation property of the duality between $C_0^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$.

From (3.4) one immediately deduces that X^* contains $\left(W_0^{1,p}(\Omega; W)\right)'$ and that the corresponding embedding

$$\left(W_0^{1,p}(\Omega; W)\right)' \hookrightarrow X^*$$

is continuous and dense. The density follows from

$$C_0^\infty(\Omega) \subset \left(W_0^{1,p}(\Omega; W)\right)' \subset X^* \subset \mathcal{D}'(\Omega). \tag{7.8}$$

Therefore, denoting by X^{**} the (strong) dual space of X^* with respect to the duality $\langle \cdot\, , \cdot\rangle$, we observe that X^{**} is continuously embedded into $W^{1,p}(\Omega; W)$ and that $C_0^\infty(\Omega)$ is weak-star dense in X^{**}. It is noteworthy that the separability of X^* in the norm topology implies that the weak-star topology on any bounded subset of X^{**} is metrizable (Rudin [33, Theorem 3.16, p. 70]). Now consider the *bipolar* (or *second conjugate*) functional to Q_V defined by

$$Q_V^{**}(u) := \sup_{f\in X^*} [\langle u, f\rangle - Q_V^*(f)] \qquad u \in X^{**}. \tag{7.9}$$

From (7.5) it is evident that

$$0 \le Q_V^{**}(u) \le Q_V(u) \quad \text{for every } u \in C_0^\infty(\Omega). \tag{7.10}$$

Moreover, in analogy with the norm $\|\cdot\|_*$ on X^* (see (7.7)), the dual norm $\|\cdot\|_{**}$ on X^{**} is given by

$$\|u\|_{**} = (p\, Q_V^{**}(u))^{1/p} . \tag{7.11}$$

The Fenchel-Young and Hölder inequalities, (7.5) and (7.6), respectively, remain valid with $Q_V^{**}(u)$ in place of Q_V. In particular, we have

$$|\langle u, f\rangle| \le \|u\|_{**}\, \|f\|_* \le (p\, Q_V(u))^{1/p}\, \|f\|_* \quad \text{for } f \in X^*,\ u \in C_0^\infty(\Omega). \tag{7.12}$$

It follows [10, Ch. I, §4], that for all $f \in X^*$,

$$\inf_{u\in C_0^\infty(\Omega)} [Q_V(u) - \langle u, f\rangle] = \inf_{u\in C_0^\infty(\Omega)} [Q_V^{**}(u) - \langle u, f\rangle] = -Q_V^*(f). \tag{7.13}$$

Definition 7.1. Given a distribution $f \in X^*$, we say that a function $u_0 \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *generalized* (or *relaxed*) *minimizer* for the functional $u \mapsto Q_V(u) - \langle u, f\rangle$ if it has the following three properties:

(i) u_0 is a (true) minimizer for the functional
$$u \mapsto Q_V^{**}(u) - \langle u, f\rangle : X^{**} \to \mathbb{R},$$
hence, $u_0 \in X^{**}$.

(ii) u_0 satisfies equation $Q'_V(u) = f$ in the sense of distributions on Ω.

(iii) There exists a (minimizing) sequence $\{u_k\}_{k=1}^{\infty} \subset C_0^\infty(\Omega)$ such that $Q_V(u_k) - \langle u_k, f\rangle \to -Q_V^*(f)$ and
$$Q'_V(u_k) - f \equiv Q'_V(u_k) - \langle \cdot\,, f\rangle \to 0 \quad \text{strongly in } W^{-1,p'}_{\mathrm{loc}}(\Omega) \tag{7.14}$$
as $k \to \infty$, together with $u_k \to u_0$ strongly in $W^{1,p}_{\mathrm{loc}}(\Omega)$, and $u_k \overset{*}{\rightharpoonup} u_0$ weakly-star in X^{**} as $k \to \infty$.

We can now formulate the existence result.

Theorem 7.2 ([36, Theorem 4.3])**.** *Let $\Omega \subset \mathbb{R}^d$ be a domain, $1 < p < \infty$, and $V \in L^\infty_{\mathrm{loc}}(\Omega)$. Assume that the nonnegative functional Q_V is weakly coercive. Then, for every $f \in X^*$, the functional $u \mapsto Q_V(u) - \langle u, f\rangle$ is bounded from below on $C_0^\infty(\Omega)$ and has a generalized minimizer u_0 in X^{**} $(\subset W^{1,p}_{\mathrm{loc}}(\Omega))$. In particular, this minimizer verifies the equation $Q'_V(u) = f$.*

8. Criticality theory

In this section we discuss positivity properties of the functional Q from [28, 30] along the lines of criticality theory for second-order linear elliptic operators [22, 23]. We note that Theorem 8.6 and Example 8.7 are new results.

Proposition 8.1. *Let $V_i \in L^\infty_{\mathrm{loc}}(\Omega)$. If $V_2 \gneqq V_1$ and $Q_{V_1} \geq 0$, then Q_{V_2} is subcritical (weakly coercive).*

Proposition 8.2. *Let $\Omega_1 \subset \Omega_2$ be domains in $\mathbb{R}^d$ such that $\Omega_2 \setminus \overline{\Omega_1} \neq \emptyset$. Let Q_V be defined on $C_0^\infty(\Omega_2)$.*

1. *If $Q_V \geq 0$ on $C_0^\infty(\Omega_2)$, then Q_V is subcritical in Ω_1.*
2. *If Q_V is critical in Ω_1, then Q_V is nonpositive in Ω_2.*

Proposition 8.3. *Let $V_0, V_1 \in L^\infty_{\mathrm{loc}}(\Omega)$, $V_0 \neq V_1$. For $t \in \mathbb{R}$ we denote*
$$Q_t(u) := tQ_{V_1}(u) + (1-t)Q_{V_0}(u), \tag{8.1}$$
and suppose that $Q_{V_i} \geq 0$ on $C_0^\infty(\Omega)$ for $i = 0, 1$.

Then $Q_t \geq 0$ on $C_0^\infty(\Omega)$ for all $t \in [0, 1]$. Moreover, Q_t is subcritical in Ω for all $t \in (0, 1)$.

Proposition 8.4. *Let Q_V be a subcritical functional in Ω. Consider $V_0 \in L^\infty(\Omega)$ such that $V_0 \not\geq 0$ and $\operatorname{supp} V_0 \Subset \Omega$. Then there exist $\tau_+ > 0$ and $-\infty \leq \tau_- < 0$ such that Q_{V+tV_0} is subcritical in Ω for $t \in (\tau_-, \tau_+)$, and $Q_{V+\tau_+V_0}$ is critical in Ω.*

Proposition 8.5. *Assume that Q_V admits a ground state v in Ω. Consider $V_0 \in L^\infty(\Omega)$ such that $\operatorname{supp} V_0 \Subset \Omega$. Then there exists $0 < \tau_+ \leq \infty$ such that Q_{V+tV_0} is subcritical in Ω for $t \in (0, \tau_+)$ if and only if*

$$\int_\Omega V_0 |v|^p \, \mathrm{d}x > 0. \tag{8.2}$$

In Propositions 8.4 and 8.5 we assumed that the perturbation V_0 has a compact support. In the following we consider a wider class of perturbations. Assume that Q is subcritical in Ω and $d > p$. It follows from Theorem 3.4 that there exists a continuous weight function W such that the following Hardy-Sobolev-Maz'ya inequality is satisfied

$$Q(u) \geq \left(\int_\Omega W(x) |u(x)|^{p^*} \, \mathrm{d}x \right)^{p/p^*} \qquad \forall u \in C_0^\infty(\Omega), \tag{8.3}$$

where $p^* = pd/(d-p)$ is the critical Sobolev exponent.

The following theorem shows that for a certain class of potentials $\tilde{V}$ the above Hardy-Sobolev-Maz'ya inequality is preserved with the same weight function W. This theorem extends the analogous result for $p = 2$ proved in [30, Theorem 2.9]. We may say that such $\tilde{V}$ are *small perturbations* of the functional Q in Ω.

Theorem 8.6. *Let Q be the functional* (1.2) *with $d > p$ and suppose that*

$$Q(u) \geq \left(\int_\Omega W |u|^{p^*} \, \mathrm{d}x \right)^{p/p^*} \qquad u \in C_0^\infty(\Omega), \tag{8.4}$$

where W is some positive continuous function (*see Theorem* 3.4*, and in particular* (3.6))*. Let*

$$\tilde{V} \in L^\infty_{\mathrm{loc}}(\Omega) \cap L^{d/p}(\Omega; W^{-d/p^*}). \tag{8.5}$$

Consider the one-parameter family of functionals $\tilde{Q}_\lambda$ defined by

$$\tilde{Q}_\lambda(u) := Q(u) + \lambda \int_\Omega \tilde{V} |u|^p \, \mathrm{d}x \qquad u \in C_0^\infty(\Omega),$$

where $\lambda \in \mathbb{R}$, and let

$$S := \{\lambda \in \mathbb{R} \mid \tilde{Q}_\lambda \geq 0 \text{ on } C_0^\infty(\Omega)\}.$$

(i) *If $\lambda \in S$ and $\tilde{Q}_\lambda$ is subcritical in Ω, then there exists $C > 0$ such that*

$$\tilde{Q}_\lambda(u) \geq C \left(\int_\Omega W |u|^{p^*} \, \mathrm{d}x \right)^{p/p^*} \qquad u \in C_0^\infty(\Omega). \tag{8.6}$$

(ii) *If $\lambda \in S$ and $\tilde{Q}_\lambda$ admits a ground state v, then for every $\psi \in C_0^\infty(\Omega)$ such that $\int_\Omega \psi v \, \mathrm{d}x \neq 0$ there exist $C, C_1 > 0$ such that the following Hardy-Sobolev-Maz'ya-Poincaré type inequality holds*

$$\tilde{Q}_\lambda(u) + C_1 \left| \int_\Omega \psi u \, \mathrm{d}x \right|^p \geq C \left(\int_\Omega W |u|^{p^*} \, \mathrm{d}x \right)^{p/p^*} \qquad u \in C_0^\infty(\Omega). \tag{8.7}$$

(iii) *The set S is a closed interval with a nonempty interior which is bounded if and only if $\tilde V$ changes its sign on a set of a positive measure in Ω. Moreover, $\lambda \in \partial S$ if and only if $\tilde Q_\lambda$ is critical in Ω.*

Proof. (i)–(ii) Assume first that $\tilde Q_\lambda$ is subcritical in Ω, and (8.6) does not hold, then there exists a sequence $\{u_k\} \subset C_0^\infty(\Omega)$ of nonnegative functions such that $\tilde Q_\lambda(u_k) \to 0$, and $\int_\Omega W|u_k|^{p^*}\,\mathrm{d}x = 1$. In light of (3.4) (with another weight function $\tilde W$) it follows that that $u_k \to 0$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$.

If $\tilde Q_\lambda$ has a ground state v, and (8.7) does not hold, then there exists a sequence $\{u_k\} \subset C_0^\infty(\Omega)$ of nonnegative functions such that $\tilde Q_\lambda(u_k) \to 0$, $\int_\Omega \psi u_k\,\mathrm{d}x \to 0$, but $\int_\Omega W|u_k|^{p^*}\,\mathrm{d}x = 1$. It follows from (3.2) (with another weight function $\tilde W$) that $u_k \to 0$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$.

Consequently, for any $K \Subset \Omega$ we have

$$\lim_{k\to\infty} \int_K |\tilde V||u_k|^p\,\mathrm{d}x = 0. \tag{8.8}$$

On the other hand, (8.5) and Hölder inequality imply that for any $\varepsilon > 0$ there exists $K_\varepsilon \Subset \Omega$ such that

$$\left|\int_{\Omega\setminus K_\varepsilon} |\tilde V||u_k|^p\,\mathrm{d}x\right| \le \left(\int_{\Omega\setminus K_\varepsilon} |\tilde V|^{d/p} W^{-d/p^*}\,\mathrm{d}x\right)^{p/d} \left(\int_\Omega W|u_k|^{p^*}\,\mathrm{d}x\right)^{p/p^*} < \varepsilon. \tag{8.9}$$

Therefore, $\int_\Omega |\tilde V||u_k|^p\,\mathrm{d}x \to 0$. Since

$$Q(u_k) \le \tilde Q_\lambda(u_k) + |\lambda| \int_\Omega |\tilde V||u_k|^p\,\mathrm{d}x, \tag{8.10}$$

it follows that $Q(u_k) \to 0$. Hence, (8.4) implies that $\int_\Omega W|u_k|^{p^*}\,\mathrm{d}x \to 0$, but this contradicts the assumption $\int_\Omega W|u_k|^{p^*}\,\mathrm{d}x = 1$. Consequently, (8.6) (resp. (8.7)) holds true.

(iii) It follows from Proposition 8.3 that S is an interval, and that $\lambda \in \operatorname{int} S$ implies that $\tilde Q_\lambda$ is subcritical in Ω. The claim on the boundedness of S is trivial and left to the reader.

On the other hand, suppose that for some $\lambda \in \mathbb{R}$ the functional $\tilde Q_\lambda$ is subcritical. By part (i), $\tilde Q_\lambda$ satisfies (8.6) with weight W. Therefore, (8.9) (with $K_\varepsilon = \emptyset$) implies that

$$\tilde Q_\lambda(u) \ge C\left(\int_\Omega W|u|^{p^*}\,\mathrm{d}x\right)^{p/p^*} \ge C_1 \left|\int_\Omega \tilde V|u|^p\,\mathrm{d}x\right| \qquad u \in C_0^\infty(\Omega). \tag{8.11}$$

Therefore, $\lambda \in \operatorname{int} S$. Consequently, $\lambda \in \partial S$ implies that $\tilde Q_\lambda$ is critical in Ω. In particular, $0 \in \operatorname{int} S$. □

Example 8.7. Let $2 \le p < d$, and let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain with a smooth boundary. Consider the Hardy functional

$$Q(u) := \int_\Omega |\nabla u|^p \,\mathrm{d}x - \left|\frac{p-1}{p}\right|^p \int_\Omega \frac{|u|^p}{d(x,\partial\Omega)^p} \,\mathrm{d}x.$$

By [11], the functional Q satisfies the following Hardy-Sobolev-Maz'ya type inequality

$$Q(u) \ge C \left(\int_\Omega |u|^{p^*} \,\mathrm{d}x\right)^{p/p^*}. \tag{8.12}$$

Let $V \in L^\infty_{\mathrm{loc}}(\Omega) \cap L^{d/p}(\Omega)$ be a positive function. By [19], there exists a constant $\lambda_* > 0$ such that

$$Q_\lambda(u) := Q(u) - \lambda \int_\Omega V|u|^p \,\mathrm{d}x$$

is nonnegative for all $\lambda \le \lambda_*$. Now, Theorem 8.6 implies that the functional $Q_{\lambda_*}(u)$ is critical in Ω. Moreover, for $\lambda < \lambda_*$ the following inequality holds

$$Q_\lambda(u) \ge C_\lambda \left(\int_\Omega |u|^{p^*} \,\mathrm{d}x\right)^{p/p^*} \qquad u \in C_0^\infty(\Omega)$$

for some $C_\lambda > 0$.

Furthermore, for every nonzero nonnegative $\psi \in C_0^\infty(\Omega)$ the following inequality holds

$$Q_{\lambda_*}(u) + C_1 \left|\int_\Omega u\psi \,\mathrm{d}x\right|^p \ge C \left(\int_\Omega |u|^{p^*} \,\mathrm{d}x\right)^{p/p^*} \qquad u \in C_0^\infty(\Omega)$$

for some $C, C_1 > 0$.

9. The linear case ($p = 2$)

Some of the results in the preceding sections have stronger counterparts in the linear case (for a recent review on the theory of positive solutions of second-order linear elliptic PDEs, see [25] and the references therein). In particular, there are several properties that are true for the linear case but are generally false or unknown in the general case. This refers in particular to SCP whose scope of validity when $p \ne 2$ is not completely understood, and to the convexity of the functional Q, which is known to be generally false for $p > 2$ (see references at the beginning of Section 7). On the other hand, as in [24, 27], we can actually consider in the linear symmetric case the following somewhat more general functional than Q_V of the form (1.2).

Let $A : \Omega \to \mathbb{R}^{d^2}$ be a measurable symmetric matrix-valued function such that for every compact set $K \Subset \Omega$ there exists $\mu_K > 1$ so that

$$\mu_K^{-1} I_d \le A(x) \le \mu_K I_d \qquad \forall x \in K, \tag{9.1}$$

where I_d is the d-dimensional identity matrix, and the matrix inequality $A \le B$ means that $B - A$ is a nonnegative matrix on $\mathbb{R}^d$. Let $V \in L^q_{\mathrm{loc}}(\Omega)$ be a real potential, where $q > d/2$. We consider the quadratic form

$$\mathbf{a}_{A,V}[u] := \frac{1}{2}\int_\Omega \left(A\nabla u \cdot \nabla u + V|u|^2\right) \mathrm{d}x \tag{9.2}$$

on $C_0^\infty(\Omega)$ associated with the Schrödinger equation

$$Pu := (-\nabla \cdot (A\nabla) + V)u = 0 \qquad \text{in } \Omega. \tag{9.3}$$

We say that $\mathbf{a}_{A,V}$ is *nonnegative* on $C_0^\infty(\Omega)$, if $\mathbf{a}_{A,V}[u] \ge 0$ for all $u \in C_0^\infty(\Omega)$.

Let v be a positive solution of the equation $Pu = 0$ in Ω. Then by [27, Lemma 2.4] we have the following analog of (2.3). For any nonnegative $w \in C_0^\infty(\Omega)$ we have

$$\mathbf{a}_{A,V}[vw] = \frac{1}{2}\int_\Omega v^2 A\nabla w \cdot \nabla w \, \mathrm{d}x. \tag{9.4}$$

Moreover, it follows from [27, 28] that all the results mentioned in this paper concerning the functional Q are also valid for the form $\mathbf{a}_{A,V}$.

9.1. Liouville-type theorem

In the linear case we have the following stronger Liouville-type statement (cf. Theorem 5.1 for $p \ne 2$).

Theorem 9.1 ([24]). *Let Ω be a domain in $\mathbb{R}^d$, $d \ge 1$. Consider two strictly elliptic Schrödinger operators with real coefficients defined on Ω of the form*

$$P_j := -\nabla \cdot (A_j\nabla) + V_j \qquad j = 0, 1, \tag{9.5}$$

where $V_j \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > d/2$, and $A_j : \Omega \to \mathbb{R}^{d^2}$ are measurable symmetric matrices satisfying (9.1).

Assume that the following assumptions hold true.

(i) *The operator P_1 admits a ground state φ in Ω.*
(ii) *$P_0 \ge 0$ on $C_0^\infty(\Omega)$, and there exists a real function $\psi \in H^1_{\mathrm{loc}}(\Omega)$ such that $\psi_+ \ne 0$, and $P_0\psi \le 0$ in Ω.*
(iii) *The following matrix inequality holds*

$$(\psi_+)^2(x)A_0(x) \le C\varphi^2(x)A_1(x) \qquad \text{a.e. in } \Omega, \tag{9.6}$$

where $C > 0$ is a positive constant.

Then the operator P_0 is critical in Ω, and ψ is the corresponding ground state. In particular, ψ is (up to a multiplicative constant) the unique positive supersolution of the equation $P_0u = 0$ in Ω.

9.2. The space $\mathcal{D}^{1,2}_{A,V}$

When $p = 2$ and the quadratic form $\mathbf{a}_{A,V}$ defined by (9.2) is nonnegative, $\mathbf{a}_{A,V}$ induces a scalar product on $C_0^\infty(\Omega)$. One can regard as the natural domain of the functional

$$\mathbf{a}^f_{A,V}(u) := \mathbf{a}_{A,V}[u] - \int_\Omega uf\,\mathrm{d}x$$

a linear space in which the functional $\mathbf{a}^f_{A,V}$ has a minimizer for all f such that $\mathbf{a}^f_{A,V}$ is bounded from below. The functional $\mathbf{a}^f_{A,V}$ is bounded from below if and only if $\int_\Omega uf\,\mathrm{d}x$ is a continuous functional with respect to the norm $(\mathbf{a}_{A,V}[u])^{1/2}$. The minimum for such f is not attained on $C_0^\infty(\Omega)$ due to the strong maximum principle, but any minimizing sequence for $\mathbf{a}^f_{A,V}$ is a Cauchy sequence. Thus, the natural domain of $\mathbf{a}_{A,V}$ is the completion of $C_0^\infty(\Omega)$ in the norm $(\mathbf{a}_{A,V}[u])^{1/2}$. In the subcritical case, due to (3.4) (which is valid also for subcritical operators of the form (9.3)), this completion is continuously imbedded into $W^{1,2}(\Omega; W)$ for some positive continuous function W. By analogy with the classical space $\mathcal{D}^{1,2}$, we denote the completion space with respect to the above norm in the subcritical case by $\mathcal{D}^{1,2}_{A,V}(\Omega)$ (see [27]).

If, however, $\mathbf{a}_{A,V}$ has a ground state v, the span of v becomes obviously the zero element of the completion space with respect to the norm $(\mathbf{a}_{A,V}[u])^{1/2}$. Recalling the definition of $\mathcal{D}^{1,2}(\mathbb{R}^d)$ for $d = 1, 2$, where the ground state of $\mathbf{a}_{I,0}[u] = \int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}x$ is $\mathbf{1}$, and in light of (3.2), we define in the critical case the norm

$$\|u\| := \left(\mathbf{a}_{A,V}[u] + \left|\int_\Omega \psi u\,\mathrm{d}x\right|^2\right)^{1/2}, \tag{9.7}$$

where ψ is any $C_0^\infty(\Omega)$-function satisfying $\int_\Omega \psi v\,\mathrm{d}x \neq 0$. Hence, also in the critical case the completion of $C_0^\infty(\Omega)$ with respect to the norm defined by (9.7) (which we also denote by $\mathcal{D}^{1,2}_{A,V}(\Omega)$) is continuously imbedded into the function space $W^{1,2}(\Omega; W)$, with an appropriate weight function W.

Example 9.2. Let $\Omega = \mathbb{R}^d = \mathbb{R}^n \times (\mathbb{R}^m \setminus \{0\})$, $1 \le m \le d$, and denote points in Ω by $(x, y) \in \mathbb{R}^n \times (\mathbb{R}^m \setminus \{0\})$. Let

$$\mathbf{a}[u] = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}y - \left(\frac{m-2}{2}\right)^2 \int_\Omega \frac{u^2}{|y|^2}\,\mathrm{d}x\,\mathrm{d}y \qquad u \in C_0^\infty(\Omega). \tag{9.8}$$

The functional (9.8) is nonnegative due to the Hardy inequality and the constant $\left(\frac{m-2}{2}\right)^2$ is the maximal constant for which this is true. Furthermore, if $m < d$, the functional (9.8) is weakly coercive, while for $m = d$ it has a generalized ground state $v(y) = |y|^{(2-m)/2}$.

It follows that completion of $C_0^\infty(\Omega)$ in the norm induced by (9.8) for $m < d$ defines a natural domain for the functional (9.8). It should be noted, however, that on the complete space the integrals in the expression (9.8) might be infinite. A more explicit characterization of the natural domain in this case can be obtained by noting that $v(x,y) = |y|^{(2-m)/2}$ is a positive solution of the corresponding equation. Thus, for the functional (9.8) the positive Lagrangian identity (2.1) gives

$$\mathbf{a}[u] = \frac{1}{2}\int_\Omega |y|^{2-m}|\nabla(|y|^{(m-2)/2}u)|^2\,\mathrm{d}x\,\mathrm{d}y. \tag{9.9}$$

It follows that the completion space can be characterized as a space of measurable functions with measurable weak derivatives for which the integral in (9.9) is finite.

An analogous definition of the natural domain for the functional Q_V could be given for general p whenever the functional Q_V is convex. We should point out that while this is in general false, one may require the convexity of another functional $\hat{Q}$, bounded by Q_V from above and from below. In particular, one can look at the functionals (2.5) or (2.6). The following statement from [30]) characterizes a convexity property of $\hat{Q}$ given by the right-hand side of (2.6).

Proposition 9.3. *Let $p > 2$, and let $v \in C^1_{\mathrm{loc}}(\Omega)$ be a fixed positive function. Then the functional*

$$\mathcal{Q}(u) := \int_\Omega \left[v^p|\nabla(u^{2/p})|^p + v^2|\nabla v|^{p-2}u^{2(p-2)/p}|\nabla(u^{2/p})|^2\right]\mathrm{d}x$$

is convex on $\{u \in C_0^\infty(\Omega), u \geq 0\}$.

9.3. Positive solutions of minimal growth

We characterize now positive solutions of minimal growth in a neighborhood of infinity of Ω in terms of a modified null sequence of the form $\mathbf{a}_{A,V}$ (cf. Section 6).

Theorem 9.4 ([29, Theorem 6.1])**.** *Suppose that $\mathbf{a}_{A,V}$ is nonnegative on $C_0^\infty(\Omega)$. Let $\Omega_1 \Subset \Omega$ be an open set, and let $u \in C(\Omega \setminus \Omega_1)$ be a positive solution of the equation $Pu = 0$ in $\Omega \setminus \overline{\Omega_1}$.*

Then $u \in \mathcal{M}_{\Omega,\overline{\Omega_1}}$ if and only if for every smooth open set Ω_2 satisfying $\Omega_1 \Subset \Omega_2 \Subset \Omega$, and an open set $B \Subset (\Omega\setminus\overline{\Omega_2})$ there exists a sequence $\{u_k\} \subset C_0^\infty(\Omega)$, $u_k \geq 0$, such that for all $k \in \mathbb{N}$, $\int_B |u_k|^2\,\mathrm{d}x = 1$, and

$$\lim_{k\to\infty}\int_{\Omega\setminus\overline{\Omega}_2} u^2 A\nabla u_k \cdot \nabla u_k\,\mathrm{d}x = 0. \tag{9.10}$$

Consider now the following Phragmén–Lindelöf-type principle that holds in unbounded or nonsmooth domains, and for irregular potential V, provided the subsolution satisfies a certain decay property (of variational type) in terms of the Lagrangian L (cf. [1, 15, 18, 31] and Section 6).

Theorem 9.5 (Comparison Principle [29]**).** *Assume that P is a nonnegative Schrödinger operator of the form (9.3). Fix smooth open sets $\Omega_1 \Subset \Omega_2 \Subset \Omega$. Let $u, v \in W^{1,p}_{\mathrm{loc}}(\Omega\setminus\Omega_1)\cap C(\Omega\setminus\Omega_1)$ be, respectively, a positive subsolution and a supersolution of the equation $Pw = 0$ in $\Omega \setminus \overline{\Omega_1}$ such that $u \leq v$ on $\partial\Omega_2$.*

Assume further that $Pu \in L^\infty_{\rm loc}(\Omega \setminus \Omega_1)$*, and that there exist an open set* $B \Subset (\Omega \setminus \overline{\Omega_2})$ *and a sequence* $\{u_k\} \subset C_0^\infty(\Omega)$*,* $u_k \geq 0$*, such that*

$$\int_B |u_k|^p \,\mathrm{d}x = 1 \quad \forall k \geq 1, \quad \text{and} \quad \lim_{k\to\infty} \int_{\Omega\setminus\overline{\Omega}_1} u^2 A\nabla(u_k/u)\cdot\nabla(u_k/u)\,\mathrm{d}x = 0. \tag{9.11}$$

Then $u \leq v$ *on* $\Omega \setminus \Omega_2$*.*

Remark 9.6. In Theorem 9.5 we assumed that the subsolution u is strictly positive. It would be useful to prove the above comparison principle under the assumption that $u \geq 0$ (cf. [15]).

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Mixed Boundary Value Problems for Stokes and Navier-Stokes Systems in Polyhedral Domains

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Dedicated to Vladimir Maz'ya on the occasion of his 70th birthday

Abstract. The paper deals with a boundary value problem for the stationary Stokes and Navier-Stokes systems, where different boundary conditions (in particular, Dirichlet, Neumann, slip conditions) are prescribed on the faces of a polyhedral domain. Various regularity results in weighted and nonweighted Sobolev and Hölder spaces are given here. Furthermore, the paper contains a maximum modulus estimate for the velocity.

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1. Introduction

It was a great pleasure and honor for me to give a plenary talk at the conference in Rome on the occasion of the 70th birthday of Vladimir Maz'ya. I became acquainted with V. Maz'ya a quarter of a century ago. In 1983, when I was Ph.D. student, I spent 8 months in Leningrad, where V. Maz'ya was my tutor. This was the beginning of a long-standing and productive cooperation, and I'm very thankful for this. In this survey article, I present some results which were obtained in the common works [23]–[27] with V. Maz'ya on mixed boundary value problems for the stationary Stokes and Navier-Stokes systems in a three-dimensional domain $\mathcal{G}$ of polyhedral type.

In the last decades, many mathematical papers appeared which treat elliptic boundary value problems in domains with piecewise smooth boundaries, for references see the books by Grisvard [8], Dauge [1], Nazarov and Plamenevskiĭ [30], Kozlov, Maz'ya and Rossmann [14]. I refer here only to papers dealing with boundary value problems for the Stokes and Navier-Stokes systems. Boundary value problems for these systems in two-dimensional polygonal domains were studied, e.g., by Kondrat'ev [12], Kellogg and Osborn [10] (Dirichlet problem), Kalex [9], Orlt and Sändig [33] (mixed boundary value problems). The Dirichlet problem for the Stokes and Navier-Stokes systems in three-dimensional domains of polyhedral type was handled in papers by Maz'ya and Plamenevskiĭ [21], Dauge [2], Nicaise [31]. Solonnikov [36, 37], Maz'ya, Plamenevskiĭ and Stupelis [22] dealt with a mixed boundary value problem for the Stokes system in a domain with nonintersecting edges, where the velocity or its normal component are prescribed on the boundary. Existence and regularity results for solutions of general elliptic boundary value problems in domains with edges and vertices are closely related to the spectral properties of certain operator pencils. For the Stokes system, these operator pencils were investigated by Maz'ya and Plamenevskiĭ [20], Dauge [2], Kozlov, Maz'ya and Schwab [17], Kozlov, Maz'ya and Rossmann [15] (see also the book [16]). Furthermore, there exists a rich theory dealing with boundary value problems in Lipschitz graph boundaries which is based on refined methods of harmonic analysis (see the survey monograph by Kenig [11]). The Dirichlet problem for the Stokes system in such domains, was studied by Fabes, Kenig and Verchota [6] who proved the existence of solutions for L_2 boundary data. Shen [34, 35] obtained an analogous result for L_p data, $2 \leq p < \infty$. Regularity results for data from Sobolev and Besov spaces were established in papers by Deuring and von Wahl [3], Mitrea and Taylor [29], Dindos and Mitrea [4], Mitrea and Monniaux [28]. Ebmeyer and Frehse [5] dealt with two special mixed boundary value problem for the Navier-Stokes system in Lipschitz domains.

In the present paper, we consider a boundary value problem for the Stokes and Navier-Stokes systems, where components of the velocity and/or the friction are prescribed on the faces of a polyhedral domain $\mathcal{G}$. To be more precise, one of the following boundary conditions is given on each face Γ_j:

(i) $u = h$,
(ii) $u_\tau = h, \quad -p + 2\nu\varepsilon_{n,n}(u) = \phi$,
(iii) $u_n = h, \quad 2\nu\varepsilon_{n,\tau}(u) = \phi$,
(iv) $-pn + 2\nu\varepsilon_n(u) = \phi$.

Here, $u_n = u \cdot n$ denotes the normal and $u_\tau = u - u_n n$ the tangential component of the velocity u, $\varepsilon(u)$ is the matrix with the elements

$$\varepsilon_{i,j}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big),$$

$\varepsilon_n(u)$ denotes the vector $\varepsilon(u)\,n$, $\varepsilon_{n,n}(u)$ is the normal component and $\varepsilon_{n,\tau}(u)$ the tangential component of $\varepsilon_n(u)$. This boundary value problem is of importance for the study of steady state flows of incompressible viscous Newtonian fluids. For example, the Dirichlet condition $u = 0$ is prescribed on solid walls. A no-friction condition (Neumann condition) $-pn + 2\nu\varepsilon_n(u) = 0$ may be useful on an artificial boundary such as the exit of a canal or a free surface. The Neumann condition appears also in the theory of hydrodynamic potentials. Condition (ii) is often used for in/out-stream surfaces, while the slip condition for uncovered fluid surfaces has the form (iii).

We are interested in regularity assertions for the variational solution

$$(u,p) \in W^{1,2}(\mathcal{G}) \times L_2(\mathcal{G})$$

in weighted and nonweighted Sobolev and Hölder spaces. Here $W^{l,s}(\mathcal{G})$ denotes the Sobolev space of all functions (or vector functions) for which the derivatives up to order l are integrable with the power s, $1 < s < \infty$. As is well known, the local regularity result $(u,p) \in W^{l,s} \times W^{l-1,s}$ is valid outside an arbitrarily small neighborhood of the edges and vertices if the data are sufficiently smooth. The same result holds for the Hölder space $C^{l,\sigma}$. Since solutions of elliptic boundary value problems in general have singularities near singular boundary points, the result cannot be globally true in $\mathcal{G}$ without any restrictions on l and s. In fact, the smoothness of the solutions depends on the eigenvalues of certain operator pencils introduced below.

The Dirichlet problem and particular mixed boundary value problems for the Stokes system can be handled in weighted Sobolev and Hölder spaces with homogeneous norms (for the Dirichlet problem see [21], the mixed boundary value problem with the boundary conditions (i) and (iii) is studied in [22]). The more general boundary value problem considered in the present paper requires the use of weighted spaces with nonhomogeneous norms. This makes the study of the boundary value problem more difficult. On the other hand, in some cases (e.g., the Dirichlet problem in a convex polyhedral domain), one gets sharper regularity results when dealing with weighted spaces with inhomogeneous norms.

2. Mixed boundary value problems for the Stokes system

In what follows, let $\mathcal{G}$ be a *domain of polyhedral type* in $\mathbb{R}^3$. This means that

- the boundary $\partial\mathcal{G}$ consists of smooth (of class C^∞) open two-dimensional manifolds Γ_j (the faces of $\mathcal{G}$), $j = 1, \dots, N$, smooth curves M_k (the edges), $k = 1, \dots, m$, and vertices $x^{(1)}, \dots, x^{(d)}$,
- for every $\xi \in M_k$ there exist a neighborhood $\mathcal{U}_\xi$ and a diffeomorphism (a C^∞ mapping) κ_ξ which maps $\mathcal{G} \cap \mathcal{U}_\xi$ onto $\mathcal{D}_\xi \cap B_1$, where $\mathcal{D}_\xi$ is a dihedron of the form $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ 0 < r < \infty, \ -\theta/2 < \varphi < \theta/2, \ x_3 \in \mathbb{R}\}$, and B_1 is the unit ball (r, φ denote the polar coordinates in the (x_1, x_2)-plane),
- for every vertex $x^{(j)}$ there exist a neighborhood $\mathcal{U}_j$ and a diffeomorphism κ_j mapping $\mathcal{G} \cap \mathcal{U}_j$ onto $\mathcal{K}_j \cap B_1$, where $\mathcal{K}_j$ is a polyhedral cone with vertex at the origin.

The set $\overline{M}_1 \cup \dots \cup \overline{M}_m$ of the singular boundary points is denoted by $\mathcal{S}$.

Let I_0, I_1, I_2, I_3 be pairwise disjoint sets such that

$$I_0 \cup I_1 \cup I_2 \cup I_3 = \{1, 2, \dots, N\}.$$

We consider the boundary value problem

$$-\nu\,\Delta u + \nabla p = f, \quad -\nabla \cdot u = g \ \text{ in } \mathcal{G}, \tag{1}$$

$$S_j u = h_j, \quad N_j(u,p) = \phi_j \ \text{ on } \Gamma_j\,, \ j = 1, \dots, N, \tag{2}$$

where (2) is one of the boundary conditions (i)–(iv), i.e., $S_j u = u$ if $j \in I_0$,

$$S_j u = u_\tau, \;\; N_j(u,p) = -p + 2\nu\varepsilon_{n,n}(u) \;\; \text{if } j \in I_1,$$
$$S_j u = u_n, \;\; N_j(u,p) = 2\nu\varepsilon_{n,\tau}(u) \;\; \text{if } j \in I_2,$$

and $N_j(u,p) = -pn + 2\nu\varepsilon_n(u)$ if $j \in I_3$. The condition $N_j(u,p) = \phi_j$ disappears for $j \in I_0$, while the condition $S_j u = h_j$ is absent for $j \in I_3$. For every index $j = 1,\dots,N$, we define the number $d_j \in \{0,1,2,3\}$ by the equality

$$d_j = k \;\; \text{if } j \in I_k.$$

Let V be the subspace of all vector functions $u = (u_1,u_2,u_3) \in W^{1,2}(\mathcal{G})$ satisfying the boundary condition $S_j u|_{\Gamma_j} = 0$ for $j = 1,\dots,N$. By a variational solution of the boundary value problem (1), (2), we mean a vector function $(u,p) \in W^{1,2}(\mathcal{G}) \times L_2(\mathcal{G})$ satisfying

$$b(u,v) - \int_{\mathcal{G}} p\, \nabla\cdot v\, dx = F(v) \;\; \text{for all } v \in V, \tag{3}$$

$$-\nabla\cdot u = g \;\text{ in } \mathcal{G}, \quad S_j u = h_j \;\text{ on } \Gamma_j, \; j = 1,\dots,N, \tag{4}$$

where

$$b(u,v) = 2\nu \int_{\mathcal{G}} \sum_{i,j=1}^{3} \varepsilon_{i,j}(u)\, \varepsilon_{i,j}(v)\, dx \tag{5}$$

and

$$F(v) = \int_{\mathcal{G}} (f + \nabla g)\cdot v\, dx + \sum_{j=1}^{n} \int_{\Gamma_j} \phi_j \cdot v\, dx. \tag{6}$$

(In the case $j \in I_1$, when ϕ_j is a scalar function, one has to replace $\phi_j \cdot v$ by $\phi_j n\cdot v$.) It is shown in [25, Theorem 5.1] that there exists a solution $(u,p) \in W^{1,2}(\mathcal{G}) \times L_2(\mathcal{G})$ of the problem (3), (4) for arbitrary $F \in V^*$, $g \in L_2(\mathcal{G})$ and $h_j \in W^{1/2,2}(\Gamma_j)$ provided g and h_j satisfy certain compatibility conditions and

$$F(v) = 0 \quad \text{for all } v \in L_V,$$

where L_V denotes the set of all $v \in V$ such that $\varepsilon_{i,j}(v) = 0$ for $i,j = 1,2,3$. Here u is unique up to elements from L_V and p is unique up to constants (unique if the boundary conditions (i) or (iii) are prescribed on at least one of the faces Γ_j).

Operator pencils generated by the boundary value problem
The smoothness of the solutions of the boundary value problem (1), (2) in a neighborhood of a singular boundary point (edge point or vertex) depends on the eigenvalues of certain operator pencils.

1) Let ξ be a point on the edge M_k, and let $\Gamma_{k_+}, \Gamma_{k_-}$ be the faces of $\mathcal{G}$ adjacent to ξ. Furthermore, let $\mathcal{D}_\xi$ be the dihedron which is bounded by the half-planes $\Gamma^\circ_{k_\pm}$ tangent to $\Gamma_{k_\pm}$ at ξ and the edge $M^\circ_\xi = \bar{\Gamma}^\circ_{k_+} \cap \bar{\Gamma}^\circ_{k_-}$. The angle between the

half-planes $\Gamma^\circ_{k_\pm}$ is denoted by θ_ξ. By r, φ, we denote the polar coordinates in the plane perpendicular to M°_ξ such that

$$\Gamma^\circ_{k_\pm} = \{x \in \mathbb{R}^3 :\ r > 0,\ \varphi = \pm\theta_\xi/2\}.$$

Then the operator $A_\xi(\lambda)$ is defined by the equality

$$A_\xi(\lambda)\,\big(U(\varphi), P(\varphi)\big) = \begin{pmatrix} r^{2-\lambda}(-\Delta u + \nabla p) \\ -r^{1-\lambda}\nabla\cdot u \\ r^{-\lambda} S_{k_\pm} u|_{\varphi=\pm\theta_\xi/2} \\ r^{1-\lambda} N_{k_\pm}(u,p)|_{\varphi=\pm\theta_\xi/2} \end{pmatrix},$$

where $u(x) = r^\lambda U(\varphi)$, $p(x) = r^{\lambda-1}P(\varphi)$, $\lambda \in \mathbb{C}$. The operator $A_\xi(\lambda)$ depends quadratically on the parameter λ and realizes a continuous mapping

$$W^{2,2}(I_\xi) \times W^{1,2}(I_\xi) \to L_2(I_\xi) \times W^{1,2}(I_\xi) \times \mathbb{C}^3 \times \mathbb{C}^3$$

for every $\lambda \in \mathbb{C}$, where I_ξ denotes the interval $(-\theta_\xi/2, +\theta_\xi/2)$. The spectrum of the pencil $A_\xi(\lambda)$ consists of eigenvalues with finite geometric and algebraic multiplicities. These eigenvalues are zeros of certain transcendental functions (see [33] and [23]). For example, for the Dirichlet and Neumann problems, the spectrum of $A_\xi(\lambda)$ consists of the solutions of the equation

$$\sin(\lambda\theta_\xi)\,\big(\lambda^2\sin^2\theta_\xi - \sin^2(\lambda\theta_\xi)\big) = 0,$$

$\lambda \neq 0$ for the Dirichlet problem.

Note that $\lambda = 1$ is always an eigenvalue of the pencil $A_\xi(\lambda)$ if $d_{k_+} + d_{k_-}$ is even. It is the eigenvalue with smallest positive real part if $\theta_\xi < \pi/m_k$, where $m_k = 1$ if $d_{k_+} + d_{k_-} \in \{0, 6\}$ and $m_k = 2$ if $d_{k_+} + d_{k_-} \in \{2, 4\}$. In this case, we denote by $\mu(\xi)$ the greatest real number such that the strip

$$0 < \operatorname{Re}\lambda < \mu(\xi) \tag{7}$$

contains only the eigenvalue $\lambda = 1$ of the pencil $A_\xi(\lambda)$. In all other cases, let $\mu(\xi)$ be the greatest real number such that the strip (7) is free of eigenvalues of the pencil $A\xi(\lambda)$. Furthermore, we put $\mu_k = \inf\limits_{\xi\in M_k} \mu(\xi)$.

2) Let $x^{(j)}$ be a vertex of $\mathcal{G}$, and let I_j be the set of all indices k such that $x^{(j)} \in \overline{\Gamma}_k$. By our assumptions, there exist a neighborhood $\mathcal{U}$ of $x^{(j)}$ and a diffeomorphism κ mapping $\mathcal{G} \cap \mathcal{U}$ onto $\mathcal{K}_j \cap B_1$ and $\Gamma_k \cap \mathcal{U}$ onto $\Gamma^\circ_k \cap B_1$ for $k \in I_j$, where

$$\mathcal{K}_j = \{x :\ x/|x| \in \Omega_j\}$$

is a polyhedral cone with vertex at the origin and $\Gamma^\circ_k = \{x :\ x/|x| \in \gamma_k\}$ are the faces of this cone. Here Ω_j is a domain of polygonal type on the unit sphere with the sides γ_k. Without loss of generality, we may assume that the Jacobian matrix $\kappa'(x)$ is equal to the identity matrix I at the point $x^{(j)}$. We introduce spherical coordinates $\rho = |x|$, $\omega = x/|x|$ in $\mathcal{K}_j$ and define

$$V_{\Omega_j} = \{u \in W^{1,2}(\Omega_j) :\ S_k u = 0 \text{ on } \gamma_k,\ k \in I_j\}.$$

Then the bilinear form $a_j(\cdot,\cdot;\lambda)$ is defined on the space $V_{\Omega_j} \times L_2(\Omega_j)$ as

$$a_j\Big(\Big(\begin{array}{c} u \\ p \end{array}\Big), \Big(\begin{array}{c} v \\ q \end{array}\Big); \lambda\Big) = \int\limits_{\substack{\mathcal{K}_j \\ 1<|x|<2}} \Big(2\nu \sum_{i,j=1}^{3} \varepsilon_{i,j}(U)\cdot \varepsilon_{i,j}(V) - P\nabla\cdot V - (\nabla\cdot U)\, Q\Big)\, dx,$$

where $U = \rho^\lambda u(\omega)$, $V = \rho^{-1-\lambda} v(\omega)$, $P = \rho^{\lambda-1} p(\omega)$, $Q = \rho^{-2-\lambda} q(\omega)$, $u, v \in V_{\Omega_j}$, $p, q \in L_2(\Omega_j)$, and $\lambda \in \mathbb{C}$. This bilinear form generates the linear and continuous operator

$$\mathfrak{A}_j(\lambda) : V_{\Omega_j} \times L_2(\Omega_j) \to V^*_{\Omega_j} \times L_2(\Omega_j)$$

by

$$\int_\Omega \mathfrak{A}_j(\lambda)\Big(\begin{array}{c} u \\ p \end{array}\Big)\cdot\Big(\begin{array}{c} v \\ q \end{array}\Big)\, d\omega = a_j\Big(\Big(\begin{array}{c} u \\ p \end{array}\Big), \Big(\begin{array}{c} v \\ q \end{array}\Big); \lambda\Big), \quad u, v \in V_{\Omega_j},\ p, q \in L_2(\Omega_j).$$

The operator $\mathfrak{A}_j(\lambda)$ depends quadratically on the complex parameter λ. The spectrum of the pencil $\mathfrak{A}_j(\lambda)$ consists of isolated points, eigenvalues with finite geometric and algebraic multiplicities. Information on the eigenvalues is given in the papers [2, 13, 17, 21] (see also the book [16]).

Regularity assertions in weighted Sobolev spaces

We denote the distance of x from the edge M_k by $r_k(x)$, the distance from the vertex $x^{(j)}$ by $\rho_j(x)$, and the distance from $\mathcal{S}$ by $r(x)$. Let X_j denote the set of the indices k such that $x^{(j)}$ is an end point of the edge M_k. Furthermore, let $\mathcal{U}_1, \dots, \mathcal{U}_d$ be domains in $\mathbb{R}^3$ such that

$$\mathcal{U}_1 \cup \dots \cup \mathcal{U}_d \supset \overline{\mathcal{G}} \quad \text{and} \quad \overline{\mathcal{U}}_j \cap \overline{M}_k = \emptyset \ \text{ if } k \notin X_j.$$

If l is a nonnegative integer, p is a real number, $p > 1$, and $\beta = (\beta_1, \dots, \beta_d)$, $\delta = (\delta_1, \dots, \delta_m)$ are tuples of real numbers, $\delta_k > -2/p$ for $k = 1, \dots, m$, then the weighted Sobolev space $W^{l,p}_{\beta,\delta}(\mathcal{G})$ is defined as the closure of the set $C_0^\infty(\overline{\mathcal{G}})$ with respect to the norm

$$\|u\|_{W^{l,p}_{\beta,\delta}(\mathcal{G})} = \Big(\sum_{j=1}^{d} \int\limits_{\mathcal{G}\cap\mathcal{U}_j} \sum_{|\alpha|\le l} \rho_j^{p(\beta_j - l + |\alpha|)} \prod_{k\in X_j} \Big(\frac{r_k}{\rho_j}\Big)^{p\delta_k} |\partial_x^\alpha u|^p\, dx\Big)^{1/p}.$$

Obviously, the space $W^{l,p}_{\beta,\delta}(\mathcal{G})$ does not depend on the choice of the domains $\mathcal{U}_j$. The trace space for $W^{l,p}_{\beta,\delta}(\mathcal{G})$ on the face Γ_j is denoted by $W^{l-1/p,p}_{\beta,\delta}(\Gamma_j)$. Furthermore, let $\mathcal{H}_{s,\beta,\delta}$ be the subspace of all vector functions $u \in W^{1,s}_{\beta,\delta}(\mathcal{G})$ such that $S_j u = 0$ on Γ_j for $j = 1, \dots, N$. The following theorem was proved in [25].

Theorem 1. *Let $(u, p) \in W^{1,2}(\mathcal{G}) \times L_2(\mathcal{G})$ be a solution of the problem* (3), (4), *where $F \in \mathcal{H}^*_{s',-\beta,-\delta}$, $g \in W^{0,s}_{\beta,\delta}(\mathcal{G})$ and $h_j \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)$. Suppose that the closed strip between the lines* $\text{Re}\lambda = -1/2$ *and* $\text{Re}\lambda = 1 - \beta - 3/s$ *does not contain eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$, $j = 1, \dots, d$, and that the components of δ satisfy* *the condition* $\max(0, 1-\mu_k) < \delta_k + 2/s < 1$ *for* $k = 1, \dots, m$. *Then* $u \in W^{1,s}_{\beta,\delta}(\mathcal{G})$ *and* $p \in W^{0,s}_{\beta,\delta}(\mathcal{G})$.

For the proof of this theorem, the authors in [25] employed point estimates for the Green's matrix of mixed boundary value problems for the Stokes system in a polyhedral cone which were obtained in [23]. The analogous higher-order regularity result in weighted Sobolev spaces holds only under additional compatibility conditions on the traces of the functions g, h_j and ϕ_j on the edges of the domain $\mathcal{G}$. A detailed description of these conditions is also given in [25].

Theorem 2. *Let* $(u, p) \in W^{1,2}(\mathcal{G}) \times L_2(\mathcal{G})$ *be a variational solution of the boundary value problem* (1), (2) *with the right-hand sides* $f \in W^{l-2,s}_{\beta,\delta}(\mathcal{G})$, $g \in W^{l-1,s}_{\beta,\delta}(\mathcal{G})$, $h_j \in W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)$, *and* $\phi_j \in W^{l-1-1/s}_{\beta,\delta}(\Gamma_j)$, $l \geq 2$. *Suppose that the closed strip between the lines* $\operatorname{Re}\lambda = -1/2$ *and* $\operatorname{Re}\lambda = l - \beta_j - 3/s$ *does not contain eigenvalues of the pencils* $\mathfrak{A}_j(\lambda)$, $j = 1, \dots, d$, *and that the components of* δ *satisfy the inequalities*

$$\max(0, l-\mu_k) < \delta_k + 2/s < l$$

for $k = 1, \dots, m$. *If moreover* g, h_j *and* ϕ_j *satisfy certain compatibility conditions on the edges* M_k, *then* $u \in W^{l,s}_{\beta,\delta}(\mathcal{G})$ *and* $p \in W^{l-1,s}_{\beta,\delta}(\mathcal{G})$.

Analogous results hold in weighted Hölder spaces (see [24]). Theorems 1 and 2 allow to obtain regularity results for the variational solutions in nonweighted Sobolev spaces if the conditions of these theorems are satisfied for $\beta = 0$ and $\delta = 0$. For this, one can use the equality $W^{l,s}_{0,0}(\mathcal{G}) = W^{l,s}(\mathcal{G})$ for $l < 3/s$. If $l > 3/s$ and $l - 3/s$ is not integer, then an arbitrary $W^{l,s}$-function is a sum of a polynomial and a $W^{l,s}_{0,0}$-function in a neighborhood of a vertex.

The maximum principle

As is known, the solution of the boundary value problem

$$-\Delta u + \nabla p = 0, \quad \nabla \cdot u = 0 \text{ in G}, \quad u|_{\partial\mathcal{G}} = h \tag{8}$$

in a domain $\mathcal{G} \subset \mathbb{R}^3$ with smooth boundary satisfies the estimate

$$\|u\|_{L_\infty(\mathcal{G})} \leq c\, \|h\|_{L_\infty(\partial\mathcal{G}} \tag{9}$$

with a constant c independent of h. This inequality was first established by Odquist [32]. A proof of this inequality can be found, e.g., in the book by Ladyzhenskaya [18]. Maz'ya and Plamenevskiĭ [21] proved the estimate (9) for solutions of the boundary value problem (8) in three-dimensional domains of polyhedral type. The proof is based on point estimates of Green's matrix. Using estimates of Green's matrix, one can also show that the solution of the problem (8) in a domain of polyhedral type satisfies the inequality

$$\sup_{x\in\mathcal{G}} d(x) \Big(\sum_{j=1}^{3} \big|\partial_{x_j} u(x)\big| + \big|p(x)\big| \Big) \leq c\, \|h\|_{L_\infty(\partial\mathcal{G}}\,,$$

where $d(x)$ denotes the distance of x to the boundary of the domain (see [27]).

3. The boundary value problem for the Navier-Stokes system

We consider the nonlinear problem

$$-\nu\,\Delta u + (u\cdot\nabla)\,u + \nabla p = f, \quad -\nabla\cdot u = g \ \text{ in } \mathcal{G}, \tag{10}$$

$$S_j u = h_j, \quad N_j(u,p) = \phi_j \ \text{ on } \Gamma_j\,,\ j = 1,\dots,N. \tag{11}$$

Using a fixed point argument, it can be easily shown that variational solutions in $W^{1,2}(\mathcal{G}) \times L_2(\mathcal{G})$ always exist if the data f, g, h_j, ϕ_j are sufficiently small and satisfy certain compatibility conditions (see [26]). In the case when the boundary conditions (ii) and (iv) disappear, such solutions exist for arbitrary f (see the books by Ladyzhenskaya [18], Temam [39], Girault and Raviart [7]).

Regularity assertions for variational solutions

In [26] it is shown that the assertion of Theorem 1 is also true for the variational solution of the boundary value problem (10), (11) if $s > 6/5$, while the assertion of Theorem 2 holds for $l = 2$ if in addition the components of β satisfy the inequality $\beta_j + 3/s < 5/2$. Using the relations between the spaces $W^{l,s}_{0,0}(\mathcal{G})$ and $W^{l,s}(\mathcal{G})$ and estimates for the eigenvalues of the pencils $A_k(\lambda)$ and $\mathfrak{A}_j(\lambda)$, one can establish a number of regularity results in nonweighted Sobolev and Hölder spaces for particular problems. In the examples below, we assume for the sake of simplicity that the data g, h_j, ϕ_j are zero.

1) Let $(u,p) \in W^{1,2}(\mathcal{G})$ be a variational solution of the *Dirichlet problem*

$$-\nu\,\Delta u + (u\cdot\nabla)\,u + \nabla p = f, \quad -\nabla\cdot u = 0 \ \text{ in } \mathcal{G}, \quad u|_{\Gamma_j} = 0\,,\ j = 1,\dots,N.$$

Then the following assertions hold.

- If $f \in W^{-1,s}(\mathcal{G})$, $2 < s \le 3$, then $(u,p) \in W^{1,s}(\mathcal{G}) \times L_s(\mathcal{G})$. If the polyhedron $\mathcal{G}$ is convex, then this assertion is true for all $s > 2$.
- If $f \in W^{-1,2}(\mathcal{G}) \cap L_s(\mathcal{G})$, $1 < s \le 4/3$, then $(u,p) \in W^{2,s}(\mathcal{G}) \times W^{1,s}(\mathcal{G})$. If $\mathcal{G}$ is convex, then this result is valid for $1 < s \le 2$. If, moreover, the angles at the edges are less than $\frac{3}{4}\pi$, then the result holds even for $1 < s < 3$.
- If $\mathcal{G}$ is convex, $f \in C^{-1,\sigma}(\mathcal{G})$, and σ is sufficiently small, then $(u,p) \in C^{1,\sigma}(\mathcal{G}) \times C^{0,\sigma}(\mathcal{G})$.

Note that the conditions on s are sharp. For special domains, it is possible to obtain precise regularity results. Let for example $\mathcal{G} = (0,2)^3 \backslash [0,1]^3$ be the difference set of two cubes. Then

$$(u,p) \in W^{1,s}(\mathcal{G}) \times L_s(\mathcal{G}) \qquad \text{if } f \in W^{-1,s}(\mathcal{G}),\ s < 4.3905\ldots,$$

$$(u,p) \in W^{2,s}(\mathcal{G}) \times W^{1,s}(\mathcal{G}) \qquad \text{if } f \in L_s(\mathcal{G}),\ s < 1.3740\ldots.$$

2) We consider a variational solution $(u,p) \in W^{1,2}(\mathcal{G}) \times L_2(\mathcal{G})$ of the *Neumann problem*

$$-\Delta u + (u\cdot\nabla)\,u + \nabla p = f, \quad -\nabla u = 0 \text{ in } \mathcal{G},$$

$$-pn + 2\nu\varepsilon_n(u) = 0 \text{ on } \Gamma_j,\ j = 1,\dots,N.$$

Suppose that $\mathcal{G}$ is a Lipschitz graph polyhedron. Then the following assertions are valid.

- If $f \in (W^{1,s'}(\mathcal{G}))^*$, $s' = s/(s-1)$, $2 < s < 3$, then $(u,p) \in W^{1,s}(\mathcal{G}) \times L_s(\mathcal{G})$.
- If $f \in (W^{1,2}(\mathcal{G}))^* \cap L_s(\mathcal{G})$, $1 < s \le 4/3$, then $(u,p) \in W^{2,s}(\mathcal{G}) \times W^{1,s}(\mathcal{G})$. If all edge angles are less than $3\arccos\frac{1}{4} \approx 1.2587\pi$, then this result is true for $1 < s < 3/2$.

3) Finally, we consider the *mixed boundary value problem*

$$-\Delta u + (u\cdot\nabla)\,u + \nabla p = f, \quad -\nabla u = 0 \text{ in } \mathcal{G},$$
$$u|_{\Gamma_j} = 0 \text{ for } j = 1,\dots,N-1, \quad u_n = 0,\ \varepsilon_{n,\tau}(u) = 0 \text{ on } \Gamma_N.$$

We suppose that the polyhedron $\mathcal{G}$ is convex and that the angle between Γ_N and the adjoining faces is less than $\pi/2$. Then the following assertions hold for every variational solution $(u,p) \in W^{1,2}(\mathcal{G}) \times L_2(\mathcal{G})$.

- If $f \in W^{-1,2}(\mathcal{G};S) \cap L_s(\mathcal{G})$, $1 < s \le 2$, then $(u,p) \in W^{2,s}(\mathcal{G}) \times W^{1,s}(\mathcal{G})$.
- If $f \in W^{-1,2}(\mathcal{G};S) \cap C^{-1,\sigma}(\mathcal{G})$ and σ is sufficiently small, then $(u,p) \in C^{1,\sigma}(\mathcal{G}) \times C^{0,\sigma}(\mathcal{G})$.

Here $W^{-1,s}(\mathcal{G};S)$ is defined as the dual space of $\{v \in W^{1,s'}(\mathcal{G}) : \ S_j v|_{\Gamma_j} = 0\}$, where $s' = s/(s-1)$.

Existence of solutions in $W^{1,s}(\mathcal{G}) \times L_s(\mathcal{G})$

In the above given results, we assumed that a variational solution $(u,p) \in W^{1,2}(\mathcal{G}) \times L_2(\mathcal{G})$ is given. Now we are interested in the question of the existence of solutions in L_s Sobolev spaces, where s may be less than 2. We give here a particular result which follows from [26, Theorem 5.2].

Theorem 3. *Let $\mathcal{G}$ be a polyhedron. We assume that one of the boundary conditions* (i)–(iii) *is given on every face Γ_j, that the Dirichlet condition is given on at least one of the adjoining faces of every edge M_k, and that $\theta_k \le 3\pi/2$ if the boundary conditions* (ii) *or* (iii) *are given on one of the adjoining faces of M_k. Furthermore, we suppose that $3/2 < s < 3$ and that the norm of $F \in W^{-1,s}(\mathcal{G};S)$ is sufficiently small. Then the problem*

$$b(u,v) + \int_{\mathcal{G}} \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j}\cdot v\,dx - \int_{\mathcal{G}} p\,\nabla\cdot v\,dx = F(v)$$
$$\textit{for all } v \in W^{1,s'}(\mathcal{G}),\ S_j v = 0 \textit{ on } \Gamma_j,$$
$$-\nabla\cdot u = 0 \ \textit{ in } \mathcal{G}, \quad S_j u = 0 \ \textit{ on } \Gamma_j,\ j = 1,\dots,N,$$

has a solution $(u,p) \in W^{1,s}(\mathcal{G}) \times L_s(\mathcal{G})$.

Here, θ_k denotes the angle at the edge M_k. For the Dirichlet problem, the assertion of Theorem 3 is valid without conditions on the angles θ_k.

A maximum modulus estimate for the velocity

For domains with smooth boundaries, it was proved by Solonnikov [38] that the solution of the problem

$$-\nu\,\Delta u + (u\cdot\nabla)\,u + \nabla p = 0, \quad -\nabla\cdot u = 0 \ \text{ in } \mathcal{G}, \quad u|_{\partial\mathcal{G}} = h \tag{12}$$

satisfies the estimate

$$\|u\|_{L_\infty(\mathcal{G})} \le c\left(\|h\|_{L_\infty(\partial\mathcal{G})}\right) \tag{13}$$

with a certain unspecified function c. An estimate of the form (13) can be also deduced from the results in Maremonti's and Russo's paper [19]. For domains of polyhedral type, it was shown in [21], that the solution u of (12) with finite Dirichlet integral is continuous in $\bar{\mathcal{G}}$ if h is continuous on $\partial\mathcal{G}$. However, the last paper does not contain estimates for the maximum modulus of u. The following theorem was proved in [27].

Theorem 4. *Let* (u,p) *be a solution of the problem* (12)*, where* $\mathcal{G}$ *is a domain of polyhedral type in* $\mathbb{R}^3$*. Then* u *satisfies the estimate* (13) *with a function* c *of the form*

$$c(t) = c_0 t\, e^{c_1 t/\nu}.$$

Here c_0 *and* c_1 *are positive constants independent of* ν*.*

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On Some Classical Operators of Variable Order in Variable Exponent Spaces

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To Vladimir Maz'ya on the occasion of his 70th birthday

Abstract. We give a survey of a selection of recent results on weighted and non-weighted estimations of classical operators of Harmonic Analysis in variable exponent Lebesgue, Morrey and Hölder spaces, based on the talk presented at International Conference *Analysis, PDEs and Applications* on the occasion of the 70th birthday of Vladimir Maz'ya, Rome, June 30–July 3, 2008. We touch both the Euclidean case and the general setting within the frameworks of quasimetric measure spaces. Some of the presented results are new.

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1. Introduction

Last decade there was a strong increase of interest to studies of various operators and function spaces in the "variable setting", when parameters defining the operator or the space (which usually are constant), may vary from point to point. A number of mathematical problems leading, for instance, to Lebesgue spaces $L^{p(\cdot)}$ with variable exponent, or Sobolev spaces $W^{m,p(\cdot)}$ arise in applications to PDE, variational problems and continuum mechanics (in particular, in the theory of the so-called electrorheological fluids), see [76]; see also a recent paper [9] on applications in the problems of image restoration. These applications stipulated a significant interest to the spaces $L^{p(\cdot)}$ in the last decade. The study of classical operators of harmonic analysis (maximal, singular operators and potential type operators) in the generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent, weighted or non-weighted, undertaken last decade, continues to attract a strong interest of researchers, influenced in particular by possible applications. We refer in particular to the surveying articles [22], [51], [82]. A progress in the study of these spaces raised a natural interest to other spaces whose parameters may be variable, for instance, Morrey spaces or Hölder (Lipschitz) spaces. The development of Harmonic Analysis and Operator Theory in the spaces $L^{p(\cdot)}$ led also to an interest to variability of parameters defining an operator.

The area which is now called variable exponent analysis, last decade became a rather branched field with many interesting results obtained in Harmonic Analysis, Approximation Theory, Operator Theory, Pseudo-Differential Operators. We present a survey of a certain selection of results on estimation of the classical operators of harmonic analysis, mainly obtained after surveys [22], [51], [82] had appeared, and present some new results on such estimations, mainly in variable Morrey and Hölder spaces. The survey is far from being complete and reflects a part of results obtained last several years. For earlier results in the topic related to Lebesgue and Sobolev spaces $L^{p(\cdot)}, W^{m,p(\cdot)}$ of variable order we refer to the above-mentioned surveys.

We start with typical examples of operators of variable order and spaces with variable exponents.

1.1. Typical examples of operators with variable orders

1^0. *The Riesz fractional integration operator* of functions on $\mathbb{R}^n$ may be considered in the case of variable order:

$$I^{\alpha(\cdot)}f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha(x)}}, \quad \alpha(x) > 0. \tag{1.1}$$

(We omit the usual normalizing constant $\frac{1}{\gamma_n(\alpha)}$; in the case where α is constant, it is for the validity of the semigroup property $I^\alpha I^\beta = I^{\alpha+\beta}$.) In general, $\alpha(x)$ may be allowed to approach singular value $\alpha(x) = 0$ at some points, and then we have to study mapping properties of $I^{\alpha(\cdot)}$ in these or other function spaces, taking into account the degeneracy of the order $\alpha(x)$.

2^0. *Hypersingular integrals.* In the case of constant α, the operator (left)-inverse to the Riesz potential operator is the fractional power $(-\Delta)^{\frac{\alpha}{2}}$ and it may be realized as a hypersingular integral, see [81]. The corresponding variable order construction (written for the case $0 < \alpha(x) < 1$) is:

$$\mathbb{D}^{\alpha(\cdot)}f(x) = \int\limits_{\mathbb{R}^n} \frac{f(x) - f(x-y)}{|y|^{n+\alpha(x)}} dy.$$

3^0. *One-dimensional Riemann-Liouville fractional integration*:

$$I^{\alpha(\cdot)}f(x) = \frac{1}{\Gamma[\alpha(x))} \int_a^x f(y)(x-y)^{\alpha(x)-1}\, dy, \quad \alpha(x) > 0,$$

as well as the corresponding fractional differentiation. Such operators have applications in Physics, see, for instance, [50].

4⁰. *Fractional operators over quasimetric measure spaces.* More generally, fractional operators of variable order may be considered on arbitrary domains in $\mathbb{R}^n$, surfaces, manifolds, fractal sets, and in general, in the setting of quasimetric measure spaces (X, d, μ) with a quasimetric d and positive Borel measure μ. It is known, see, for instance, [25], [51], [52], that they are defined in different forms, not equivalent in general,

$$\mathfrak{I}^{\alpha(\cdot)} f(x) = \int\limits_X \frac{[d(x,y)]^{\alpha(x)}}{\mu B(x, d(x,y))} f(y)\, d\mu(y),$$
$$\mathfrak{I}^{\alpha} f(x) = \int_\Omega \frac{f(y)\, d\mu(y)}{\mu B(x, d(x,y))^{1-\alpha(x)}}, \tag{1.2}$$

and

$$I^{\alpha(\cdot)} f(x) = \int\limits_X \frac{f(y)\, d\mu(y)}{[d(x,y)]^{N-\alpha(x)}}, \tag{1.3}$$

where $\alpha(x) > 0$ and N should be thought as a kind of dimension of X. However, in general, X may have no "dimension", but may have the so-called lower and upper dimensions, which in their turn may depend on the point x. In the case where the measure satisfies the growth condition $\mu B(x,r) \leq Cr^N$ with some $N > 0$, this exponent N (not necessarily an integer), may be used to define $I^{\alpha(\cdot)} f(x)$.

5⁰. *Fractional maximal function.* Another example of an operator of variable order is the fractional maximal function

$$M^{\alpha(\cdot)} f = \sup_{r>0} r^{-\alpha(x)} \int_{|y-x|<r} f(y)\, dy$$

and its corresponding version for an arbitrary quasimetric measure space.

6⁰. *Fractional powers of operators of variable order.* In general, one may also consider fractional powers $A^{\alpha(x)}$ of this or other operator A; however, different definitions of such powers, which coincide in the case $\alpha = \text{const}$, now may lead do quite different objects. We do not touch this topic here.

1.2. Typical examples of spaces with variable exponents

1. *Generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent* (see [85], [62] and surveys [22], [51], [82]) defined by the condition

$$\int_\Omega |f(x)|^{p(x)}\, dx < \infty.$$

2. More generally, *Musielak-Orlicz spaces $L^{\Phi(\cdot)}(\Omega)$ with the Young function also varying from point to point* (see [68], [21]):

$$\int_\Omega \Phi[x, f(x)]\, dx < \infty.$$

3. *Variable exponent Morrey spaces* $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ ([3], [53], [54]) defined by

$$\sup_{x\in\Omega,\ r>0} r^{-\lambda(x)} \int\limits_{B(x,r)\cap\Omega} |f(y)|^{p(y)}dy < \infty. \tag{1.4}$$

4. *Hölder spaces* $H^{\lambda(\cdot)}(\Omega)$ *of variable order* ([47], [48], [74]), defined by the condition

$$\sup_{|h|<t} |f(x+h) - f(x)| \le Ct^{\lambda(x)},\ x \in \Omega.$$

5. More generally, *generalized Hölder spaces with variable characteristic* $\omega(h) = \omega(x,h)$ depending on x ([88]):

$$\sup_{|h|<t} |f(x+h) - f(x)| \le C\omega(x,t),$$

that is, the spaces of continuous functions with a given dominant of their continuity modulus, which may vary from point to point.

Notation

(X, d, μ) is a measure space with quasimetric d and a non-negative measure μ;
$B(x, r) = B_X(x, r) = \{y \in X : d(x, y) < r\}$;
$p'(x) = \frac{p(x)}{p(x)-1}$, $1 < p(x) < \infty$, $\frac{1}{p(x)} + \frac{1}{p'(x)} \equiv 1$;
$p_- = p_-(X) = \inf\limits_{x\in X} p(x)$, $p^+ = p^+(X) = \sup\limits_{x\in X} p(x)$;
$p'_- = \inf\limits_{x\in X} p'(x) = \frac{p^+}{p^+-1}$, $(p')^+ = \sup\limits_{x\in X} p'(x) = \frac{p_-}{p_--1}$;
$\mathbb{P}(X)$, see (2.2)–(2.3);
a.i. = almost increasing $\Longleftrightarrow u(x) \le Cu(y)$ for $x \le y, C > 0$.
$\mathbb{R}^n$ is the n-dimensional Euclidean space;
Ω is a non-empty open set in $\mathbb{R}^n$ or Ω;
d_Ω denotes the diameter of Ω;
χ_E is a characteristic function of a measurable set $E \subset \mathbb{R}^n$;
$|E|$ is the Lebesgue measure of E;
by c and C we denote various absolute positive constants, which may have different values even in the same line.

2. Some basics for variable exponent Lebesgue spaces

In the sequel (X, d, μ) is a homogeneous type space, i.e., a measure space with a quasimetric d and a non-negative measure μ satisfying the doubling condition; we refer to [11], [25], [42], for the basic notions of function spaces on quasimetric measure spaces. The space (X, d, μ) is assumed to satisfy the conditions:

1) all the balls $B(x, r)$ are measurable,
2) the space $C(X)$ of uniformly continuous functions on X is dense in $L^1(\mu)$.

The doubling condition means that $\mu B(x, 2r) \le C\mu B(x, r)$.

By $L^{p(\cdot)}(X,\varrho)$, where $\varrho(x)\geq 0$, we denote the weighted Banach space of measurable functions $f: X\to\mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(X,\varrho)} := \|\varrho f\|_{p(\cdot)} = \inf\left\{\lambda>0: \int\limits_X \left|\frac{\varrho(x)f(x)}{\lambda}\right|^{p(x)} d\mu(x)\leq 1\right\}<\infty. \tag{2.1}$$

We write $L^{p(\cdot)}(X,1)=L^{p(\cdot)}(X)$ and $\|f\|_{L^{p(\cdot)}(X)}=\|f\|_{p(\cdot)}$ in the case $\varrho(x)\equiv 1$.

The generalized Lebesgue spaces $L^{p(\cdot)}(X)$ with variable exponent on quasimetric measure spaces have been considered in [31], [35], [44], [45], [46], [49], [66], the Euclidean case being studied in [26], [29], [62], [85], see also references therein.

By $\mathbb{P}(X)$ we denote the set of bounded measurable functions $p(x)$ defined on X which satisfy the condition

$$1<p_-\leq p(x)\leq p^+<\infty, \quad x\in X \tag{2.2}$$

and by $WL(X)$ we denote the set of functions $p(x)$ such that

$$|p(x)-p(y)|\leq \frac{A}{\ln\frac{1}{d(x,y)}}, \quad d(x,y)\leq\frac{1}{2}, \quad x,\, y\in X. \tag{2.3}$$

3. On some recent results on boundedness of classical operators in spaces $L^{p(\cdot)}(\Omega,\varrho)$

The boundedness of various classical operators in $L^{p(\cdot)}(\Omega)$, $\Omega\subseteq\mathbb{R}^n$, in the non-weighted case was proved in [12] by the extrapolation method. An extension to the case of weighted estimates, including the setting of quasimetric measure spaces, was given in [58], [57]. In relation to the extrapolation method, we refer to [75], [15], [16], [17].

We touch the cases not covered in [12], [58], [57] for the following operators

1) *Convolution operators*

$$Af(x)=\int\limits_{\mathbb{R}^n} k(y)f(x-y)dy \tag{3.1}$$

with rather "nice" kernels for which the local log-condition is not needed,

2) *Hardy-Littlewood maximal operator*

$$\mathcal{M}f(x)=\sup_{r>0}\frac{1}{\mu(B(x,r))}\int\limits_{B(x,r)}|f(y)|\,d\mu(y), \quad x\in X \tag{3.2}$$

where X in general is a quasimetric measure space, being either an open set in $\mathbb{R}^n$ or a Carleson curve on the complex plane in this section; we pay a special attention to this special case of metric measure spaces with constant dimension – Carleson curves – because of important application in operator theory;

3) *the Cauchy singular integral operator*

$$S_\Gamma f(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - t} d\nu(\tau) \tag{3.3}$$

along a Carleson curve Γ on complex plane, where ν is the arc-length measure;

4) *potential type operators.*

3.1. On convolution operators

We single out a result on convolution operators, obtained without local log-condition. As is known, the Young theorem in its natural form is not valid in the case of variable exponent, whatsoever smooth exponent $p(x)$ is. As observed by L. Diening, the Young theorem is valid under the log-condition on $p(x)$ if the kernel is dominated by a radial integrable non-increasing function. However, a natural expectation was that the Young theorem may be valid in the case of rather "nice" kernels without the local log-condition, which was proved in [23], see Theorem 3.1.

Let $\mathcal{P}_\infty(\mathbb{R}^n)$ be the set of measurable bounded functions on $\mathbb{R}^n$ such that $1 \leq p_- \leq p(x) \leq p_+ < \infty, \quad x \in \mathbb{R}^n,$ there exists $p(\infty) = \lim\limits_{x\to\infty} p(x)$ and

$$|p(x) - p(\infty)| \leq \frac{A}{\ln(2+|x|)}, \quad x \in \mathbb{R}^n. \tag{3.4}$$

Theorem 3.1. *Let* $|k(y)| \leq C(1+|y|)^{-\lambda}, y \in \mathbb{R}^n$ *for some* $\lambda > n\left(1 - \frac{1}{p(\infty)} + \frac{1}{q(\infty)}\right)$. *Then the operator* (3.1) *is bounded from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to* $L^{q(\cdot)}(\mathbb{R}^n)$ *under the only assumption that* $p, q \in \mathcal{P}_\infty(\mathbb{R}^n)$ *and* $q(\infty) \geq p(\infty)$.

3.2. On the maximal operator

Let

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)|\, d\mu(y) \tag{3.5}$$

be the Hardy-Littlewood maximal operator. In the case of constant $p \in (1,\infty)$ the boundedness of the maximal operator on bounded quasimetric measure spaces is well known, due to A.P. Calderón [8] and R. Macías and C. Segovia [65], for weights in the Muckenhoupt class $A_p = A_p(X)$. For variable exponents, the non-weighted boundedness of the maximal operator was first proved in the Euclidean case in [18], [19], for bounded domains or for $\mathbb{R}^n$ with $p(x) \equiv$ const outside some large ball. For further results in non-weighted case see [13], [14], [21], [63], [69], [70]. Extensions to the case of quasimetric measure spaces were considered in [45] and [49].

Let, by definition, $A_{p(\cdot)}(X)$:= Muckenhoupt class be the class of weights for which the maximal operator is bounded in the space $L^{p(\cdot)}(X, \varrho)$. By $\widetilde{A}_{p(\cdot)}(X)$ we denote the class of weights, which satisfy the "Muckenhoupt-like looking" condition

$$\sup_{x\in X, r>0}\left(\frac{1}{\mu B(x,r)}\int\limits_{B(x,r)}|\varrho(y)|^{p(y)}d\mu(y)\right)\left(\frac{1}{\mu B(x,r)}\int\limits_{B(x,r)}\frac{d\mu(y)}{|\varrho(y)|^{\frac{p(y)}{p_- -1}}}\right)^{p_- -1}<\infty, \tag{3.6}$$

where $p_- = \inf_{x\in X} p(x)$. The class $\widetilde{A}_{p(\cdot)}(X)$ coincides with $A_{p(\cdot)}(X)$ in the case where p is constant. The next theorem ([61] in the case X is a Carleson curve and [60], [59] in the general case) states that $\widetilde{A}_{p(\cdot)}(X) \subset A_{p(\cdot)}(X)$ under natural conditions.

Theorem 3.2. *Let X be a bounded doubling measure quasimetric space. Under conditions* (2.2)*,* (2.3) *and* (3.6)*, $\mathcal{M}$ is bounded in $L^{p(\cdot)}(X,\varrho)$.*

In the case of power weights or radial type weights, the boundedness of the maximal operator was obtained under conditions weaker than derived from (3.6). We refer for details to [58], [57], but mention that for a radial weight $w(|x-a|)$, $a\in\Omega$, with w in the so-called Bary-Stechkin type class, the condition on the weight, in the Euclidean case, reduces to

$$-\frac{n}{p(a)} < m(w) \le M(w) < \frac{n}{p'(a)} \tag{3.7}$$

in terms of the Matuszewska-Orlicz indices $m(w)$ and $M(w)$ of the function $w(r)$; see a version of (3.7) for quasimetric measure spaces in [58], [57]. The sufficiency of the above condition in terms of the numbers $m(w)$ and $M(w)$ seems to be a new result even in the case of constant p. In relation with (3.7), note that in applications, the verification of the Muckenhoupt condition for a concrete weight may be an uneasy task, even in the case of constant p. Therefore, it is always of importance to have easier sufficient conditions for weight functions, as, for instance, in (3.7).

3.3. On the Cauchy singular operator

We specially dwell on the case of the Cauchy singular operator along Carleson curves because of of its importance in application to singular integral equations.

Theorem 3.3 ([55])**.** *Let Γ be a simple Carleson curve of finite or infinite length, let $p\in\mathbb{P}(\Gamma)\cap WL(\Gamma)$ and the following condition at infinity*

$$|p(t)-p(\tau)| \le \frac{A_\infty}{\ln\frac{1}{\left|\frac{1}{t}-\frac{1}{\tau}\right|}}, \qquad \left|\frac{1}{t}-\frac{1}{\tau}\right| \le \frac{1}{2},$$

for $|t|\ge L$, $|\tau|\ge L$ with some $L>0$, in the case Γ is infinite. Then the operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma,\varrho)$ with weight $\varrho(t) = (1+|t|)^\beta \prod_{k=1}^{m}|t-t_k|^{\beta_k}$, $t_k\in\Gamma$, *if and only if*

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \qquad k = 1, \ldots, m, \qquad \textit{and} \qquad -\frac{1}{p(\infty)} < \beta + \sum_{k=1}^{m} \beta_k < \frac{1}{p'(\infty)}, \tag{3.8}$$

the latter condition appearing in the case Γ *is infinite.*

An extension of Theorem 3.3 to the case of radial type oscillating weights from the Zygmund-Bary-Stechkin class Φ_δ^β may be found in [56]. The following is an extension of the Guy David theorem to the case of variable exponent $p(x)$.

Theorem 3.4. *Let* Γ *be a finite rectifiable curve and* $p : \Gamma \to [1, \infty)$ *a continuous function. If the operator* S_Γ *is bounded in* $L^{p(\cdot)}(\Gamma)$, *then the curve* Γ *has the property*

$$\sup_{\substack{t\in\Gamma \\ r>0}} \frac{\nu(\Gamma \cap B(t,r))}{r^{1-\varepsilon}} < \infty \tag{3.9}$$

for every $\varepsilon > 0$. *If* $p(t)$ *satisfies the log-condition* (2.3), *then* (3.9) *holds with* $\varepsilon = 0$, *i.e.,* Γ *is a Carleson curve.*

3.4. On potential operators

For non-weighted results on potentials and Sobolev embeddings we refer to [12], [20], [24], [27], [28], [34], [67], [80].

a) Weighted $p(\cdot) \to q(\cdot)$-boundedness. The known generalization of Sobolev theorem by Stein-Weiss for the case of power weights was extended in [83], [84] to the variable exponent setting as follows.

Theorem 3.5. *Let* $p \in \mathbb{P}(\mathbb{R}^n) \cap WL(\mathbb{R}^n)$, $\sup\limits_{x\in\mathbb{R}^n} p(x) < \frac{n}{\alpha}$, $\varrho(x) = |x|^{\gamma_0}(1+|x|)^{\gamma_\infty - \gamma_0}$ *and*

$$|p_*(x) - p_*(y)| \le \frac{A_\infty}{\ln \frac{1}{|x-y|}}, \qquad |x-y| \le \frac{1}{2}, \quad x, y \in \mathbb{R}^n, \quad p_*(x) = p\left(\frac{x}{|x|^2}\right). \tag{3.10}$$

Then operator (1.1) *with* $\alpha(x) = \alpha = \mathrm{const}$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ *to* $L^{q(\cdot)}(\mathbb{R}^n, \varrho)$, *if*

$$\alpha - \frac{n}{p(0)} < \gamma_0 < \frac{n}{p'(0)}, \qquad \alpha - \frac{n}{p(\infty)} < \gamma_\infty < \frac{n}{p'(\infty)}. \tag{3.11}$$

We refer to [78] for a generalization of Theorem 3.5 to the case of more general radial type weights. In connection with estimation of operators over unbounded domains, we refer also to a certain general approach suggested in [40].

b) Characterization of the range of potential operators. The inversion of the Riesz potentials with densities in $L^{p(\cdot)}(\mathbb{R}^n)$ by means of hypersingular integrals

$$\mathbb{D}^\alpha f = \frac{1}{d_{n,\ell}(\alpha)} \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{\left(\Delta_y^\ell f\right)(x)}{|y|^{n+\alpha}}\, dy, \tag{3.12}$$

known also as Riesz fractional derivatives of order α, was obtained in [2] (we refer to [81] for the case of constant p and hypersingular integrals in general). This gave a possibility to obtain in [6] a characterization of the range $I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ in terms of convergence of $\mathbb{D}^\alpha f$ in $L^{p(\cdot)}(\mathbb{R}^n)$ as follows.

Theorem 3.6. *Let $p \in WL(\mathbb{R}^n)$, $1 < p_-(\mathbb{R}^n) \le p^+(\mathbb{R}^n) < \frac{n}{\alpha}$ and f a locally integrable function. Then $f \in I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$, if and only if $f \in L^{q(\cdot)}(\mathbb{R}^n)$ with $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, and $\mathbb{D}^\alpha f \in L^{p(\cdot)}$ (treated in the sense of convergence in $L^{p(\cdot)}$).*

A study of the range $I^\alpha[L^{p(\cdot)}(\Omega)]$ for domains $\Omega \subset \mathbb{R}^n$ is an open question; in the form given in Theorem 3.6 it is open even in the case of constant p, one of the reasons being in the absence of the corresponding apparatus of hypersingular integrals adjusted to domains in $\mathbb{R}^n$; some their analogue reflecting the influence of the boundary was recently suggested in [72] for the case $0 < \alpha < 1$. In the one-dimensional case for $\Omega = (a, b), -\infty < a < b \le \infty$, when the range of the potential coincides with that of the Riemann-Liouville fractional integral operators (in the case $1 < p^+ < \frac{1}{\alpha}$), the characterization for variable $p(x)$ was obtained in [73], where for $-\infty < a < b < \infty$ there was also shown its coincidence with the space of restrictions of Bessel potentials.

The result of Theorem 3.6 was used in [6] to obtain a characterization of the *Bessel potential space*

$$\mathcal{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = \{f : \ f = \mathcal{B}^\alpha\varphi, \quad \varphi \in L^{p(\cdot)}(\mathbb{R}^n)\}, \quad \alpha \ge 0,$$

where $\mathcal{B}^\alpha\varphi = F^{-1}(1 + |\xi|^2)^{-\alpha/2}F\varphi$. It runs as follows.

Theorem 3.7. *Under the conditions of Theorem* 3.6

$$\mathcal{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = L^{p(\cdot)}(\mathbb{R}^n)\bigcap I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = \{f \in L^{p(\cdot)}(\mathbb{R}^n) : \ \mathbb{D}^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n)\} \tag{3.13}$$

and $\mathcal{B}^m[L^{p(\cdot)}(\mathbb{R}^n)] = W^{m,p(\cdot)}(\mathbb{R}^n)$ for any integer $m \in \mathbb{N}_0$, where $W^{m,p(\cdot)}(\mathbb{R}^n)$ is the Sobolev space with the variable exponent $p(x)$.

Statement (3.13) has the following generalization, see [73], Theorem 4.10.

Theorem 3.8. *Let $\mathrm{Y} = \mathrm{Y}(\mathbb{R}^n)$ be a Banach function space, satisfying the assumptions*

i) *C_0^∞ is dense in Y;*
ii) *the maximal operator $\mathcal{M}$ is bounded in Y;*
iii) *$I^\alpha f(x)$ converges absolutely for almost all x for every $f \in \mathrm{Y}$ and $(1 + |x|)^{-n-\alpha} I^\alpha f(x) \in L^1(\mathbb{R}^n)$.*

Then

$$\mathcal{B}^\alpha(\mathrm{Y}) = \mathrm{Y}\bigcap I^\alpha(\mathrm{Y}) = \{f \in \mathrm{Y} : \mathbb{D}^\alpha f = \lim_{\substack{\varepsilon\to 0\\(\mathrm{Y})}} \mathbb{D}^\alpha_\varepsilon f \in \mathrm{Y}\}. \tag{3.14}$$

From Theorem 3.8 there follows, in particular, the characterization of the ranges of potential operators over weighted Lebesgue spaces with variable exponent obtained by means of results of Subsection 3.2 for the maximal operator.

Certain results related to imbedding of the range of the Riesz potential operator into Hölder spaces (of variable order) in the case $p(x) \geq n$ were obtained in [7]. The results proved in [7] run as follows. In Theorem 3.9 we use the notation $\Pi_{p,\Omega} := \{x \in \Omega : p(x) > n\}$ and $H^{\alpha(\cdot)}(\Omega)$ in Theorem 3.15 is the Hölder-type space with a finite seminorm $[f]_{\alpha(\cdot),\Omega} := \sup_{x,x+h\in\Omega, 0<|h|\leq 1} |h|^{-\alpha(x)}|f(x) - f(x+h)|$.

Theorem 3.9. *Let Ω be a bounded open set with Lipschitz boundary, let $p \in WL(\Omega), p_+(\Omega) < \infty$ and let the set $\Pi_{p,\Omega}$ be non-empty. If $f \in W^{1,p(\cdot)}(\Omega)$, then*

$$|f(x) - f(y)| \leq C(x,y) \, \||\nabla f|\|_{p(\cdot),\Omega} \, |x-y|^{1-\frac{n}{\min[p(x),p(y)]}} \tag{3.15}$$

for all $x, y \in \Pi_{p,\Omega}$ such that $|x-y| \leq 1$, where $C(x,y) = \frac{c}{\min[p(x),p(y)]-n}$ with $c > 0$ not depending on f, x and y.

Theorem 3.10. *Let Ω be a bounded open set with Lipschitz boundary, $p \in WL(\Omega)$ and $p_+(\Omega) < \infty$. If $\inf\limits_{x\in\Omega} p(x) > n$, then $W^{1,p(\cdot)}(\Omega) \hookrightarrow H^{1-\frac{n}{p(\cdot)}}(\Omega)$.*

Theorem 3.10 is an improved version of the result earlier obtained in [27], [30]. The papers [32], [33] are also relevant to the topic. We refer also to [43] where the capacity approach was used to get embeddings into the space of continuous functions or into $L^\infty(\Omega)$. In [7] there were also obtained $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}$-estimates of hypersingular integrals (fractional differentiation operators)

$$\mathcal{D}^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(x) - f(y)}{|x-y|^{n+\alpha(x)}} \, dy, \quad x \in \Omega. \tag{3.16}$$

We dwell briefly also on extensions to the case of Hajłasz-Sobolev spaces on quasimetric measure spaces. In [4], by means of the estimate

$$|f(x) - f(y)| \leq C(\mu, \alpha, \beta) \, \left[d(x,y)^{\alpha(x)} \mathcal{M}^\sharp_{\alpha(\cdot)} f(x) + d(x,y)^{\beta(y)} \mathcal{M}^\sharp_{\beta(\cdot)} f(y)\right]$$

where $\mathcal{M}^\sharp_{\alpha(\cdot)} f(x) = \sup\limits_{r>0} \frac{r^{-\alpha(x)}}{\mu B(x,r)} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \, d\mu(y)$, generalizing an estimate from [41], there was given an extension of (3.15) with n replaced by the exponent from the growth condition and ∇f replaced by the generalized gradient of f. This led to the following result for the Hajłasz-Sobolev space $M^{1,p(\cdot)}(X)$.

Theorem 3.11. *Let the set X be bounded and the measure μ be doubling. If $p(\cdot)$ is* log-*Hölder continuous and $p_- > N$, then $M^{1,p(\cdot)}(X) \hookrightarrow H^{1-\frac{N}{p(\cdot)}}(X)$.*

We refer also to [5] with regards to Sobolev-type estimations with variable $p(\cdot)$ of potentials $\mathfrak{J}^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ on metric measure spaces.

4. Maximal and potential operators in variable exponent Morrey spaces

In this section we present results obtained for maximal and potential operators in variable exponent Morrey spaces.

$$I^{\alpha(\cdot)}f(x) = \int_\Omega \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}}$$

of variable order $\alpha(x)$. We prove the boundedness of the maximal operator in Morrey spaces under the log-condition on $p(\cdot)$. For potential operators, under the same log-condition and the assumptions $\inf_{x\in\Omega}\alpha(x) > 0,\ \sup_{x\in\Omega}[\lambda(x)+\alpha(x)p(x)] < n$, we present a Sobolev type $L^{p(\cdot),\lambda(\cdot)} \to L^{q(\cdot),\lambda(\cdot)}$-theorem. In the case of constant α, we also give a result on the boundedness theorem in the limiting case $p(x) = \frac{n-\lambda(x)}{\alpha}$, when the potential operator I^α acts from $L^{p(\cdot),\lambda(\cdot)}$ into BMO.

Let $p(\cdot)$ and $\lambda(\cdot)$ be measurable functions on $\Omega \subseteq \mathbb{R}^n$ with values in $[0,n]$. We define the variable Morrey space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ by condition (1.4). Equipped with the norm

$$\|f\| = \inf\left\{\eta > 0 : \sup_{x\in\Omega,\ r>0} r^{-\lambda(x)} \int\limits_{B(x,r)\cap\Omega} \left(\frac{|f(y)|}{\eta}\right)^{p(y)} dy \le 1\right\}$$
$$= \sup_{x\in\Omega,\ r>0} \left\| r^{-\frac{\lambda(x)}{p(\cdot)}} f\,\chi_{\widetilde{B}(x,r)}\right\|_{p(\cdot)},$$

this is a Banach space. In the case where $|\Omega| < \infty$ and $\lambda(\cdot)$ is log-continuous, this norm is equivalent to $\sup_{x\in\Omega,\ r>0}\left\| r^{-\frac{\lambda(\cdot)}{p(\cdot)}} f\,\chi_{\widetilde{B}(x,r)}\right\|_{p(\cdot)}$. There holds the embedding $L^{q(\cdot),\mu(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot),\lambda(\cdot)}(\Omega)$, when $\frac{n-\lambda(x)}{p(x)} \ge \frac{n-\mu(x)}{q(x)}$.

In the sequel we suppose that $0 \le \lambda(x) \le \lambda_+ < n, \quad x \in \Omega$. For constant exponents $p(x) \equiv p$ and $\lambda(x) \equiv \lambda$ the following two theorems were proved in [10], [1], respectively.

Theorem 4.1. *Let Ω be bounded and $p \in \mathbb{P}(\Omega)\cap WL(\Omega)$. Then the maximal operator M is bounded in the space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$.*

Theorem 4.2. *Let Ω be bounded, $p \in \mathbb{P}(\Omega) \cap WL(\Omega)$ and $\alpha \in WL(\Omega)$. Under the conditions $\inf\limits_{x\in\Omega}\alpha(x) > 0, \quad \sup_{x\in\Omega}[\lambda(x)+\alpha(x)p(x)] < n$, the operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^{q(\cdot),\lambda(\cdot)}(\Omega)$, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n-\lambda(x)}$.*

For the limiting case $p(x) = \frac{n-\lambda(x)}{\alpha(x)}$, we have the following statement, proved in [71], Theorem 5.4, for constant exponents.

Theorem 4.3. *Let $0 < \alpha < n$, $\lambda(x) \ge 0$, $\sup_{x\in\Omega}\lambda(x) < n-\alpha$, $\lambda \in WL(\Omega)$ and let $p(x) = \frac{n-\lambda(x)}{\alpha}$. Then the operator I^α is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $BMO(\Omega)$.*

Theorem 4.3 is derived – via the pointwise estimate $M^\sharp(I^\alpha f)(x) \le c\,M^\alpha f(x)$, ([1], Proposition 3.3) – from the following statement.

Theorem 4.4. *Let Ω be bounded, $p \in \mathbb{P}(\Omega) \cap WL(\Omega)$ and $\inf_{x\in\Omega} \alpha(x) > 0$. In the case $p(x) = \frac{n-\lambda(x)}{\alpha}$, the fractional maximal operator*

$$M^{\alpha(\cdot)} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha(x)}{n}}} \int\limits_{\widetilde{B}(x,r)} |f(y)| dy$$

is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^{\infty}(\Omega)$.

5. Fractional integrals and hypersingular integrals in variable order Hölder spaces on homogeneous spaces

The results we present here are new, they were obtained in [77]. For a version of such results when X is a sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, we refer to [79]. We consider the mapping properties of potential type operators in Hölder spaces $H^{\lambda(\cdot)}$ of variable order, which in case of domains Ω in $\mathbb{R}^n$ are defined by the condition $\sup_{|h|<t} |f(x+h)-f(x)| \leq Ct^{\lambda(x)}$, $x \in \Omega$. It is done in the general setting of quasimetric measure spaces (X, d, μ) which satisfy the growth condition

$$\mu B(x,r) \leq K r^N \quad \text{as} \quad r \to 0, \quad K > 0, \tag{5.1}$$

where $N > 0$ need not be an integer, for the potentials of form (1.3), where we admit variable exponent $\alpha(x)$, $0 \leq \alpha(x) < 1$, and Ω is an open bounded set in a quasimetric measure space X. We will also study the corresponding hypersingular operators

$$(D^{\alpha} f)(x) = \lim_{\varepsilon\to 0} \int\limits_{y\in\Omega:\varrho(x,y)>\varepsilon} \frac{f(y)-f(x)}{\varrho(x,y)^{N+\alpha(x)}} d\mu(y), \quad x \in \Omega, \tag{5.2}$$

within the frameworks of the Hölder spaces $H^{\lambda(\cdot)}(\Omega)$ with a variable exponent. In the case of constant α such a study in the general setting of quasimetric measure spaces (X, ϱ, μ) with growth condition, is known, see [36], [37], [38], [39].

The estimate we present here reveal the mapping properties of the operators I^{α} and D^{α} in dependence of local values of $\alpha(x)$ and $\lambda(x)$. Note that estimations with variable $\lambda(x)$ and $\alpha(x)$ were known in the special case $X = \mathbb{S}^{n-1}$ for spherical potential operators and related hypersingular integrals, and even in a more general setting of generalized Hölder spaces defined by a given (variable) dominant $w(x,h)$ of continuity modulus, see, for instance, [86], [87], [89].

The estimates we present here are related to a general quasimetric measure spaces and admit the situation when $\alpha(x)$ may be degenerate on Ω (on a set of measure zero). We denote

$$\Pi_{\alpha} = \{x \in \Omega : \alpha(x) = 0\}$$

and suppose that $\mu(\Pi_{\alpha}) = 0$.

To obtain results stating that the range of the potential operator over this or that Hölder space is imbedded into a better space of a similar nature, we prove Zygmund type estimates for the continuity modulus. In the case we study, these estimates are local, depending on points x. By means of such Zygmund type estimates of such a kind, we prove theorems on the mapping properties $I^{\alpha(\cdot)}$: $H^{\lambda(\cdot)}(\Omega) \to H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$, and similar results for the operator $D^{\alpha(\cdot)}$, $0 < \alpha(x) < 1$.

Let (X, d, μ) be a homogeneous quasimetric measure space. As shown in [64], it admits an equivalent quasimetric d_1 for which there exists an exponent $\theta \in (0, 1]$ such that the property

$$|d_1(x,z) - d_1(y,z)| \le M d_1^{\theta}(x,y) \left\{ d_1(x,z) + d_1(y,z) \right\}^{1-\theta} \tag{5.3}$$

holds. When d is a metric, then d automatically satisfies (5.3) with $\theta = 1$ and $M = 1$. For brevity, we will say that the quasimetric d is *regular of order* $\theta \in (0, 1]$, if it satisfies property (5.3).

In the sequel we suppose that all the balls $B(x,r) = \{y \in X : d(x,y) < r\}$ are measurable and $\mu S(x,r) = 0$ for all the spheres $S(x,r) = \{y \in X : d(x,y) = r\}, \quad x \in X \quad r \ge 0$.

For fixed $x \in \Omega$ we consider the local continuity modulus

$$\omega(f,x,h) = \sup_{\substack{z\in\Omega: \\ d(x,z)\le h}} |f(x) - f(z)| \tag{5.4}$$

of a function f at the point x. Everywhere below we assume that $|h| < 1$. The function $\omega(f,x,h)$ is non-decreasing in h and tends to zero as $h \to +0$ for any continuous function on Ω and fixed x.

Lemma 5.1. *For all $x, y \in \Omega$ such that $d(x,y) \le h$, the inequality*

$$\frac{1}{C}\omega(f,x,h) \le \omega(f,y,h) \le C\omega(x,y,h) \tag{5.5}$$

holds, where $C = [2k] + 2$ and k is the constant from the triangle inequality. If $a(x) \in WL(\Omega)$, then

$$\frac{1}{C} h^{a(x)} \le h^{a(y)} \le C h^{a(x)} \tag{5.6}$$

for all x, y such that $d(x,y) < h$, where $C \ge 1$ depends on the function a, but does not depend on x, y and h.

For a function $\lambda(x)$ defined on Ω we suppose that

$$\lambda_- := \inf_{x\in X} \lambda(x) > 0 \qquad \text{and} \quad \lambda_+ := \sup_{x\in X} \lambda(x) < 1.$$

Definition 5.2. *By $H^{\lambda(\cdot)}(\Omega)$ we denote the space of functions $f \in C(\Omega)$ such that $\omega(f,x,h) \le Ch^{\lambda(x)}$, where $C > 0$ does not depend on $x, y \in \Omega$. Equipped with the norm*

$$\|f\|_{H^{\lambda(\cdot)}(\Omega)} = \|f\|_{C(\Omega)} + \sup_{x\in\Omega} \sup_{h\in(0,1)} \frac{\omega(f,x,h)}{h^{\lambda(x)}},$$

this is a Banach space.

In Hölder norm estimations of functions $I^\alpha f$, the case $f \equiv \text{const}$ plays an important role, in the case where

$$\mathfrak{I}_\alpha(x) := I^\alpha(1)(x) = \int\limits_\Omega \frac{d\mu(z)}{d(x,z)^{N-\alpha(x)}} \tag{5.7}$$

is well defined. Observe that in the Euclidean case $\Omega = X = \mathbb{R}^N$, this integral although not well directly defined, may be treated as a constant in the case $\alpha(x) = \alpha = \text{const}$ in the sense that the cancellation property

$$\int\limits_{\mathbb{R}^N} \left[\frac{1}{|z-x|^{N-\alpha}} - \frac{1}{|z-y|^{N-\alpha}} \right] dz \equiv 0, \quad 0 < \alpha < 1, \quad x, y \in \mathbb{R}^N$$

holds. For constant α, the function $\mathfrak{I}_\alpha(x)$ is also constant in the case $\Omega = X = \mathbb{S}^{N-1}$, which fails when $\alpha = \alpha(x)$ and the cancellation property of the type

$$\int\limits_\Omega \left[\frac{1}{|z-x|^{N-\alpha(x)}} - \frac{1}{|z-y|^{N-\alpha(y)}} \right] d\mu(z) \equiv 0,$$

no more holds even for $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{S}^{N-1}$; see, for instance, [36] on the importance of the cancellation property $I^\alpha(1) \equiv \text{const}$ for the validity of mapping properties of potentials within Hölder spaces on quasimetric measure spaces. When we consider Hölder type spaces $H^{\lambda(\cdot)}(\Omega)$ which contain constants, the condition

$$\mathfrak{I}_\alpha(1) \in H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$$

is necessary for the mapping $I^\alpha : H^{\lambda(\cdot)}(\Omega) \to H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$ to hold.

Remark 5.3. Let $\inf\limits_{x\in\Omega} \alpha(x) \geq 0$ and $x, y \notin \Pi_\alpha$. Then

$$|\mathfrak{I}_\alpha(x) - \mathfrak{I}_\alpha(y)| \leq C \frac{|\alpha(x)-\alpha(y)|}{\min(\alpha(x),\alpha(y))} + \left| \int\limits_\Omega \left[d(x,z)^{\alpha(x)-N} - d(y,z)^{\alpha(x)-N} \right] d\mu(z) \right| \tag{5.8}$$

and

$$\begin{aligned} &|\alpha(x)\mathfrak{I}_\alpha(x) - \alpha(y)\mathfrak{I}_\alpha(y)| \\ &\quad \leq C\,|\alpha(x)-\alpha(y)| + \min(\alpha(x),\alpha(y)) \left| \int\limits_\Omega \left[d(x,z)^{\alpha(x)-N} - d(y,z)^{\alpha(x)-N} \right] d\mu(z) \right| \end{aligned} \tag{5.9}$$

where $C > 0$ does not depend on $x, y \in \Omega$.

Remark 5.4. The meaning of estimates (5.8)–(5.9) is in the fact that the second term on the right-hand sides may be subject to the cancellation property: at the least it disappears when $\Omega = X = \mathbb{R}^N$ or $\Omega = X = \mathbb{S}^{N-1}$.

The estimate (5.10) provided by the following theorem clearly shows the worsening of the behaviour of the local continuity modulus $\omega(I^\alpha f, x, h)$ when x approaches the points where $\alpha(x)$ vanishes. We also give a weighted estimate exactly with the weight $\alpha(x)$.

We use the notation

$$\alpha_h(x) = \min_{d(x,y)<h} \alpha(y).$$

Theorem 5.5. *Let Ω be a bounded open set in X, let $\alpha \in WL(\Omega)$ and $0 \leq \inf_{x\in\Omega} \alpha(x) \leq \sup_{x\in\Omega} \alpha(x) < \min(1, N)$. Then for all the points $x \in \Omega\backslash\Pi_\alpha$ such that $\alpha_h(x) \neq 0, 0 < h < \frac{d}{2}$, the following Zygmund type estimate is valid*

$$\omega(I^\alpha f, x, h) \leq \frac{C}{\alpha_h(x)} h^{\alpha(x)} \omega(f, x, h) + Ch^\theta \int\limits_h^d \frac{\omega(f, x, t)dt}{t^{1+\theta-\alpha(x)}} \tag{5.10}$$

$$+ C\omega(\alpha, x, h) \int\limits_h^d \frac{\omega(f, x, t)dt}{t^{2-\alpha(x)}} + C\omega(\mathfrak{I}_\alpha, x, h)\|f\|_{C(\Omega)},$$

where the constant $C > 0$ does not depend on f, x and h.

Also, for all the points $x \in \Omega\backslash\Pi_\alpha$ the weighted estimate holds

$$\omega(\alpha I^\alpha f, x, h) \leq Ch^{\alpha(x)} \omega(f, x, h) + Ch^\theta \int\limits_h^d \frac{\omega(f, x, t)dt}{t^{1+\theta-\alpha(x)}} \tag{5.11}$$

$$+ C\omega(\alpha, x, h) \int\limits_h^d \frac{\omega(f, x, t)dt}{t^{2-\alpha(x)}} + C\omega(\alpha\mathfrak{I}_\alpha, x, h)\|f\|_{C(\Omega)},$$

5.1. Zygmund type estimates of hypersingular integrals

Theorem 5.6. *Let $\alpha \in WL(\Omega)$ and $0 \leq \inf_{x\in\Omega} \alpha(x) \leq \max_{x\in\Omega} \alpha(x) < 1$. If $f \in C(\Omega)$, then for all $x, y \in \Omega$ with $d(x, y) < h$ such that $\alpha(x) \neq 0$ and $\alpha(y) \neq 0$, the following estimate is valid*

$$|(D^\alpha f)(x) - (D^\alpha f)(y)| \leq \frac{C}{\min(\alpha(x), \alpha(y))} \int\limits_0^h \left[\frac{\omega(f, x, t)}{t^{1+\alpha(x)}} + \frac{\omega(f, y, t)}{t^{1+\alpha(y)}}\right] dt \tag{5.12}$$

$$+ C \int\limits_h^2 \left[\omega(\alpha, x, h) + h^\theta t^{1-\theta}\right] \frac{\omega(f, x, t)dt}{t^{2+\alpha(x)}},$$

where $C > 0$ does not depend on x, y and h.

5.2. Theorems on mapping properties

Recall that for the potential operator $I^{\alpha(\cdot)}$ we allow the variable order $\alpha(x)$ to be degenerate on a set Π_α (of measure zero). We consider the weighted space

$$H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega, \alpha) = \{f : \alpha(x)f(x) \in H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega\}.$$

Theorem 5.7. *Let* $\alpha(x) \geq 0, \quad \max_{x\in\Omega} \alpha(x) < \min(\theta, N), \quad \alpha(x) \in \mathrm{Lip}(\Omega)$, *and*

$$\sup_{x\in\Omega}[\lambda(x) + \alpha(x)] < \theta. \tag{5.13}$$

If

$$\alpha\mathfrak{I}_\alpha \in H^{\lambda(\cdot)+\alpha(\cdot)}, \tag{5.14}$$

then the operator $I^{\alpha(\cdot)}$ *is bounded from the space* $H^{\lambda(\cdot)}(\Omega)$ *into the weighted space* $H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega, \alpha)$.

A "non-degeneracy" version of Theorem 5.7 obtained from (5.10), runs as follows.

Theorem 5.8. *Let*

$$0 < \min_{x\in\Omega} \alpha(x) \leq \max_{x\in\Omega} \alpha(x) < \min(\theta, N) \quad \textit{and} \quad \alpha \in \mathrm{Lip}(\Omega). \tag{5.15}$$

Under conditions (5.13) *and the condition* $\mathfrak{I}_\alpha \in H^{\lambda(\cdot)+\alpha(\cdot)}$, *the operator* $I^{\alpha(\cdot)}$ *is bounded from the space* $H^{\lambda(\cdot)}(\Omega)$ *into the space* $H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$.

The corresponding mapping theorem for the hypersingular operator runs as follows.

Theorem 5.9. *Under conditions* (5.14), (5.15), *the operator* $D^{\alpha(\cdot)}$ *is bounded from the space* $H^{\lambda(\cdot)}(\Omega)$ *into the space* $H^{\lambda(\cdot)-\alpha(\cdot)}(\Omega)$, *if*

$$0 < \inf_{x\in\Omega}\{\lambda(x) - \alpha(x)\}, \quad \sup_{x\in\Omega} \lambda(x) < 1.$$

Added in proofs. After the paper was submitted, the problem of Muckenhoupt weights for variable exponent Lebesgue spaces, touched in Subsection 3.2, in the Euclidean setting was solved by P. Hästö and L. Diening, "Muckenhoupt weights in variable exponent spaces", preprint available at
http://www.helsinki.fi/~pharjule/varsob/publications.shtml

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Irregular Conductive Layers

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Dedicated to Professor Vladimir G. Maz'ya on His 70th birthday

Abstract. In this survey some singular homogenization results are described. This approach leads to the spectral convergence of a sequence of weighted second-order elliptic partial differential operators to a singular elliptic operator with a fractal term.

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1. Introduction

I consider, in this survey, a two-dimensional elastic membrane which has been reinforced with the inclusion of a lower-dimensional elastic strip. The strip is thin, but its conductivity is large. I study the asymptotic behavior of such a composite medium, when the thickness of the strip vanishes and the conductivity of the strip becomes infinite, the product of thickness and conductivity remaining bounded. A peculiar feature of the model is that the strips develop, in the limit, infinite length and fractal geometry.

Singular homogenization problems of this kind go back to the work of Pham Huy and Sanchez Palencia [31], Cannon and Meyer [4] in the 1970's and of Attouch [1] in the 1980's. These authors consider a 3-dimensional domain with the inclusion of a flat, or smooth, 2-dimensional elastic layer. More general examples of asymptotic singular limits have been studied by U. Mosco, [24], in the framework of Dirichlet forms and variational convergence.

In this survey, the inclusion can be a fractal von Koch curve, a mixture of von Koch curves, and also a Sierpiński curve. Both the von Koch and the Sierpiński curve have infinite length. They have, however, different Hausdorff dimensions, moreover, the spectral dimension of the von Koch string is 1, while the Sierpiński membrane has a spectral dimension intermediate between 1 and 2. As we shall see, the asymptotic behavior of the composite medium – consisting of the initial membrane with the inclusion of the fractal membrane – is deeply affected by these dimensional parameters.

The fractal model considered in this paper is inspired by the problems first considered in [19] on so-called *second-order transmission problems* with infinitely conductive fractal layers. The problems studied in [19] display a peculiar feature. They consist in two equations, both of second order, which are coupled by a second-order transmission equation on the layer. Taken together with the boundary and transmission conditions, the two equations describe the variational equilibrium of the composite medium, as it results from the interaction of the full-dimensional Euclidean dynamics with the lower-dimensional fractal dynamics of the layer.

The approach of this survey provides additional physical motivation to the problems considered in [19] and in [20] that can be obtained as the limit – in the homogenization sense – of the equilibrium equations of fully-dimensional membranes reinforced by the inclusion of suitable thin strips. The singular fractal (or pre-fractal) membrane occurs only in the limit, as the asymptotic effect of the physical characteristics of the strips.

Before describing the results in more detail, we note that Dirichlet forms with singular energy measures have been studied by Tomisaki, Ikeda-Watanabe and Fukushima ([32], [18] and [6]). These authors also describe the singular Markov processes associated with these forms. More recently, Markov processes associated with Dirichlet forms with singular fractal energies have been studied by Lindstrøm, Jonsson and Kumagai (see [22], [11] and [15]). However, these studies do not address the homogenization problem consisting in getting the singular forms as the limit of non-singular full-dimensional energies.

Our model consists in a polygonal domain Ω of the plane, in which we insert a fractal set. We see three examples: the von Koch curve K, a mixture of Koch curves $K^{(\xi)}$ and the Sierpiński gasket $\mathcal{G}$. The fractal is constructed by iteration of N contractive similarities and K^n, $K^{(\xi)}_n$, G^n denote the pre-fractal curve obtained after n-iterations of the contractive similarities generating the von Koch curve K, the mixture of Koch curves $K^{(\xi)}$ and the Sierpiński gasket $\mathcal{G}$ respectively. Then each side of polygonal curve (K^n, $K^{(\xi)}_n$, G^n) is replaced by a thin hexagonal polygon with two opposite vertices at the end points of that side and the remaining four vertices at the vertices of a thin rectangle parallel to the side and with transversal size ε (see the red polygonal strips in Figure 2, Figure 3, Figure 5 and Figure 6). In this way we obtain a polygonal strip surrounding the pre-fractal curve, which will be denoted by Σ^n_ε. The elastic characteristics of the strip Σ^n_ε are incorporated in the definition of a *weight function* w^n_ε supported on Σ^n_ε. The function w^n_ε is singular at the intersections of two successive sides of the pre-fractal curve. Analytically, the weight w^n_ε belongs to the Muckenhoupt class A_2 with respect to the two-dimensional Lebesgue measure on $\mathbb{R}^2$. The full-dimensional dynamics is described – for fixed n and ε – by the singular elliptic operator

$$A^n_\varepsilon u = -\mathrm{div}(a^n_\varepsilon \nabla u) \tag{1.1}$$

where the coefficient a_ε^n is taken to be equal to 1 on $\Omega - \Sigma_\varepsilon^n$ and equal to $\sigma_n w_\varepsilon^n$ on Σ_ε^n, the scalar coefficients $\sigma_n > 0$ being chosen appropriately. The sequence of operators A_ϵ^n converges to a self-adjoint operator A in $L^2(\Omega)$. The limit operator A turns out to be the generator in $L^2(\Omega)$ of the energy form E obtained as the M-limit of the energy forms E_ε^n of the operators A_ε^n, as $n \to +\infty$. The form E is the sum of an energy term supported on Ω and a singular energy term supported on the fractal set:

$$E[u] = \int_\Omega |\nabla u|^2 dxdy \, + \, \mathcal{E}[u|] \tag{1.2}$$

where $u|$ denotes the trace of u on the fractal set.

The bilinear form $\mathcal{E}(\cdot,\cdot)$ associated with the energy $\mathcal{E}$ is a (regular) Dirichlet form with (dense) domain $D_0[\mathcal{E}]$ in the space $L^2(\cdot,\mu)$, where μ is the normalized invariant measure in the fractal. Fractal Dirichlet forms of this kind were first constructed by Kusuoka and Fukushima-Shima, ([17], [8]) who extended early probabilistic constructions of Barlow-Perkins, Lindstrøm, Goldstein and Kusuoka ([3],[21],[9] and [16]) as well as finite-difference analytic definitions of Kigami ([13]). The full limit form (1.2) is itself a (regular) Dirichlet form in the space $L^2(\Omega)$, with domain

$$D_0[E] = \{u : \, u \in H_0^1(\Omega), u| \in D_0[\mathcal{E}]\}\,. \tag{1.3}$$

The M-convergence of the energy forms E_ε^n can be characterized in terms of convergence of the resolvent operators, semigroups and spectral families associated with the forms allowing developments and applications (see Theorem 2.4.1, Corollaries 2.6.1 and 2.7.1 of [24]). However in this survey, I will not deal with these consequences of Theorems 2.2, 3.1 and 4.1.

We conclude this introduction by briefly describing the structure of the paper. Section 2 concerns the von Koch curve (see Theorem 2.2), Section 3 concerns mixtures of Koch curves (see Theorem 3.1) and Section 4 concerns the Sierpiński gasket (see Theorem 4.1).

2. The von Koch curve

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ say the "rectangle" $[0,1]\times[-1,1]$.
Let $\alpha \in (2,4]$ and $\vartheta = \arcsin\sqrt{\alpha - \frac{\alpha^2}{4}} \in [0, \frac{\pi}{2})$.
We consider the 4 contractive similarities $\{\psi_1,\psi_2,\psi_3,\psi_4\}$ in $\mathbb{R}^2$

$$\psi_1(z) = \frac{z}{\alpha}, \;\; \psi_2(z) = \frac{z}{\alpha}\, e^{i\vartheta} + \frac{1}{\alpha}, \;\; \psi_3(z) = \frac{z}{\alpha}\, e^{-i\vartheta} + \frac{1}{2} + \frac{i\sin\vartheta}{\alpha}, \;\; \psi_4(z) = \frac{z+\alpha-1}{\alpha},$$

$(z \in \mathbb{C})$.

For each integer $n > 0$ we consider arbitrary n-tuples of indices $i|n = (i_1, i_2, \ldots, i_n) \in \{1,2,3,4\}^n$. We define

$$\psi_{i|n} = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_n}, \tag{2.1}$$

and for any $\mathcal{O}(\subseteq \mathbb{R}^2)$ we set $\mathcal{O}^{i|n} = \psi_{i|n}(\mathcal{O})$.

The von Koch curve K can be constructed as the closure (in $\mathbb{R}^2$) of the set (of points) V^∞ where:

$$V_0 = \{A, B\}, \quad V^n = \bigcup_{i|n} V_0^{i|n} \quad \text{and} \quad V^\infty = \bigcup_{n=0}^{+\infty} V^n \tag{2.2}$$

and

$$A = (0,0) \;\; B = (1,0). \tag{2.3}$$

The von Koch curve K is a d-set (in the sense of [12]) with respect to the Hausdorff measure $\mathcal{H}^d, d = \ln 4/\ln\alpha$. Moreover, an energy form $\mathcal{E}[u]$ is also defined on K, which is the limit of an increasing sequence of quadratic forms constructed by finite difference schemes, namely

$$\begin{cases} \mathcal{E}[u] = \lim_{n\to+\infty} \mathcal{E}^{(n)}[u] \\ \mathcal{E}^{(n)}[u] = \rho^n \sum_{i|n} (u(\psi_{i|n}(A)) - u(\psi_{i|n}(B)))^2 \end{cases} . \tag{2.4}$$

where $\rho = 4$. The associated bilinear form $\mathcal{E}(\cdot,\cdot)$ is a regular Dirichlet form on $L^2(K, \mathcal{H}^d)$, with a domain $D[\mathcal{E}]$ dense in $L^2(K, \mathcal{H}^d)$. The functions $u \in D[\mathcal{E}]$ turn out to be continuous functions on K, which are indeed Hölder continuous with exponent $\delta = d/2$. The subspace of $D[\mathcal{E}]$ of all functions $u \in D[\mathcal{E}]$, that vanish at the end-points A and B of K, will be denoted by $D_0[\mathcal{E}]$. In the following, we consider the form $\mathcal{E}$ always on its domain $D_0[\mathcal{E}]$. For definitions and more details on these properties, we refer to [7], [16] and [19].

In particular we are dealing with the polygonal curve K^n that is the "pre-fractal" curve at the n-generation approximating the generalized Koch curve K:

$$K^n = \bigcup_{i|n} K_0^{i|n} \tag{2.5}$$

where K_0 is the segment of end-points $A = (0,0)$ and $B = (1,0)$ that can be considered as the initial curve, *i.e.,* at the 0-step of the iteration procedure.

Let me note that the geometrical aspect of pre-fractal Koch curve depends on the contraction factor $\alpha \in (2,4]$ as we see in Figure 1 where we have chosen the contraction factor $\alpha = 3$, $\alpha = 2,2$ and $\alpha = 3,8$ respectively.

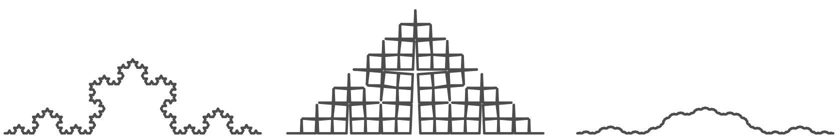

FIGURE 1. Pre-fractal Koch curves with different contraction factors

For every $0 < \varepsilon \le \varepsilon_0 \le \frac{c_1}{2}$, where $c_1 = 2\tan\frac{\beta}{4}$, $\beta = \min\{\pi - 2\vartheta, \vartheta\}$, we define the ε-neighborhood of K_0, denoted by Σ_ε, to be the "open" polygonal domain whose vertices are the points A, P_1, P_2, B, P_3, P_4 where

$$P_1 := \left(\frac{\varepsilon}{c_1}, \frac{\varepsilon}{2}\right), \quad P_2 := \left(1-\frac{\varepsilon}{c_1}, \frac{\varepsilon}{2}\right), \quad P_3 := \left(1-\frac{\varepsilon}{c_1}, -\frac{\varepsilon}{2}\right), \quad P_4 := \left(\frac{\varepsilon}{c_1}, -\frac{\varepsilon}{2}\right).$$

We then divide Σ_ε into three parts: the rectangle $\mathcal{R}_\varepsilon$ of vertices P_1, P_2, P_3, P_4 and the two triangles $\mathcal{T}_{j,\varepsilon}$, $j = 1, 2$ of vertices A, P_1, P_4 and P_2, B, P_3, (respectively in the order). For every n and ε, as above, we define the ε-neighborhood, Σ_ε^n, of K^n to be the (open) polygonal domain

$$\Sigma_\varepsilon^n = \bigcup_{i|n} \Sigma_\varepsilon^{i|n} \quad \text{and} \quad \Sigma_\varepsilon^{i|n} = \psi_{i|n}(\Sigma_\varepsilon). \tag{2.6}$$

Note that Σ_ε^n is a topological neighborhood of $K^n \setminus V^n$.

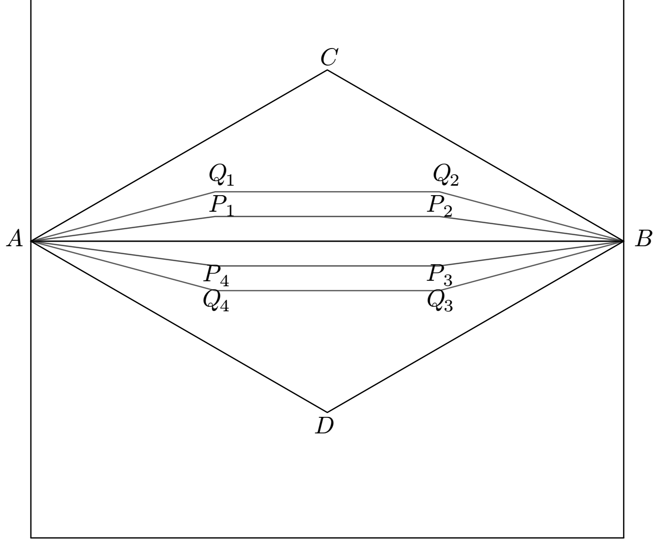

FIGURE 2. Geometry of the layer

In the domain Ω, for given n and ε, we now define a *weight* w_ε^n as follows: let P belong to the boundary $\partial(\Sigma_\varepsilon)^{i|n}$ of $\Sigma_\varepsilon^{i|n}$ denote by $P^\perp$ the "orthogonal" projection of P on $K_0^{i|n}$ and by $|P - P^\perp|$ the (Euclidean) distance between P and $P^\perp$ (in $\mathbb{R}^2$). If (x, y) belongs to the segment of end points P and $P^\perp$ we set, in our current notation,

$$w_\varepsilon^n(x,y) = \begin{cases} \frac{2+c_1^2}{4|P-P^\perp|} & \text{if} \quad (x,y) \in \mathcal{T}_{j,\varepsilon}^{i|n}, \quad j = 1,2 \\ \frac{1}{2|P-P^\perp|} & \text{if} \quad (x,y) \in \mathcal{R}_\varepsilon^{i|n}. \end{cases} \tag{2.7}$$

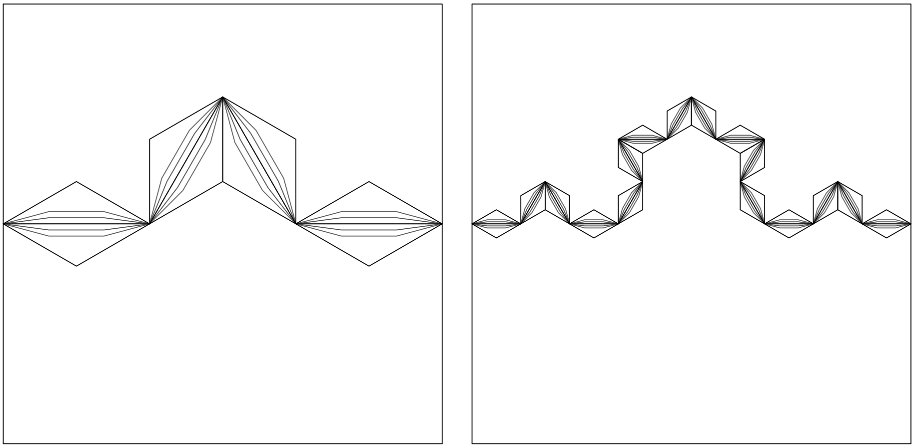

FIGURE 3. “Iterated” layers

Moreover we set:

$$w_\varepsilon^n(x,y) = 1 \qquad \text{if} \quad (x,y) \notin \Sigma_\varepsilon^n. \tag{2.8}$$

Associated with the weight w_ε^n are the Sobolev spaces

$$H^1(\Omega; w_\varepsilon^n) = \left\{ u \in L^2(\Omega) : \int_\Omega |\nabla u|^2 w_\varepsilon^n dxdy < +\infty \right\} \tag{2.9}$$

and $H_0^1(\Omega; w_\varepsilon^n)$, the latter being the completion of $C_0^\infty(\Omega)$ in the norm:

$$\|u\|_{H^1(\Omega;w_\varepsilon^n)} := \left\{ \int_\Omega |u|^2 dxdy + \int_\Omega |\nabla u|^2 w_\varepsilon^n dxdy \right\}^{\frac{1}{2}}$$

and the “weighted” energy functionals F_ε^n in $L^2(\Omega)$

$$F_\varepsilon^n[u] = \begin{cases} \int_\Omega a_\varepsilon^n(x,y)|\nabla u|^2 dxdy & if \quad u \in H_0^1(\Omega; w_n^\varepsilon) \\ +\infty & if \quad u \in L^2(\Omega) \setminus H_0^1(\Omega; w_n^\varepsilon) \end{cases} \tag{2.10}$$

where the unbounded conductivity coefficient is

$$a_\varepsilon^n(x,y) = \begin{cases} \sigma_n w_\varepsilon^n(x,y) & if \quad (x,y) \in \Sigma_\varepsilon^n \\ 1 & if \quad (x,y) \notin \Sigma_\varepsilon^n. \end{cases} \tag{2.11}$$

In order to state the results, we also need to recall the notion of M-convergence of functionals, introduced in [23], see also [24].

Definition 2.1. A family of functional F_ε M-converges to a functional F in $L^2(\Omega)$ if

(a) For every v_ε converging weakly to u in $L^2(\Omega)$

$$\underline{\lim} F_\varepsilon[v_\varepsilon] \geq F[u], \quad as \quad \varepsilon \to 0. \tag{2.12}$$

(b) For every $u \in L^2(\Omega)$ there exists u_ε^* converging strongly in $L^2(\Omega)$ such that

$$\overline{\lim} F_\varepsilon[u_\varepsilon^*] \leq F[u], \quad as \quad \varepsilon \to 0. \tag{2.13}$$

We can now state the first convergence result, with notation as given above:

Theorem 2.2. *Let $N = 4$, $\alpha \in \mathbb{Q}$, $\rho = 4$. For every $n \in \mathbb{N}$, let $\varepsilon_n = (\rho/N)^n \omega_n$, where $\omega_n \leq c_1/2$ is an arbitrary sequence such that $\omega_n \to 0$ as $n \to +\infty$, and let $\sigma_n = (\rho/\alpha)^n$. Then the sequence of functionals $F^n_{\varepsilon_n}$ defined in* (2.10) *M-converges, as $n \to +\infty$, to the functional F in $L^2(\Omega)$ where F is the extended value energy functional in $L^2(\Omega)$*

$$F[u] = \begin{cases} \int_\Omega |\nabla u|^2 dxdy + \mathcal{E}[u|_K] & \text{if} \quad u \in D_0[E] \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus D_0[E], \end{cases}$$

where $D_0[E] = \{H^1_0(\Omega), u|_K \in D_0[\mathcal{E}]\}$.
Here $u|_K$ denotes the trace of u on the fractal set K.

For the proof of Theorem 2.2, comments and details I refer to [28] and also to [5].

3. Koch mixtures

The mathematical theory of fractal bodies has been based mainly on self-similarity. Strict self-similarity, however, is a too stringent property to be realistic in physical applications. This has led to the investigation of more general models which can be seen as deterministic or random "mixtures" of self similar fractals.

Environment dependent fractals are generated by families of Euclidean similarities operating in random way that mimics the influence of the environment. The limit mixture and all relevant analytic estimates depend on the asymptotic frequency of the occurrence of each family. Harmonic functions on certain random scale-irregular fractals exhibit an interesting analytic behavior; they are continuous functions, however uniform Harnack inequalities on decreasing balls do not hold for them: in order to be global energy minimizers, they are forced by the complicated fine geometry of the body to develop locally very sharp oscillations at every point (see [2] and [27]).

In this section I discuss the case of a deterministic mixture of two Koch curves (see [33], [27] and [26] for mixtures of Sierpiński gaskets).

Consider the families of the contractive similarities defined in previous Section 2

$$\psi_1^{(\alpha)}(z) = \frac{z}{\alpha}, \quad \psi_2^{(\alpha)}(z) = \frac{z}{\alpha}\, e^{i\vartheta} + \frac{1}{\alpha},$$

$$\psi_3^{(\alpha)}(z) = \frac{z}{\alpha}\, e^{-i\vartheta} + \frac{1}{2} + \frac{i \sin\vartheta}{\alpha}, \quad \psi_4^{(\alpha)}(z) = \frac{z + \alpha - 1}{\alpha}$$

and fix two different values of the contraction factor $2 < \alpha_1 < \alpha_2 < 4$. The two families of the relative contractive similarities in $\mathbb{R}^2$ are denoted by

$$\Psi^{(1)} = \left\{\psi_1^{(1)}, \psi_2^{(1)}, \psi_3^{(1)}, \psi_4^{(1)}\right\}, \quad \Psi^{(2)} = \left\{\psi_1^{(2)}, \psi_2^{(2)}, \psi_3^{(2)} \psi_4^{(2)}\right\}. \tag{3.1}$$

Let ξ be the sequence $\xi = (\xi_1, \xi_2, \xi_3, \ldots), \xi_i \in \{1,2\}$. The mixture of two Koch curves $K^{(\xi)}$ can be constructed as the closure (in $\mathbb{R}^2$) of the set (of points) $V_\infty^{(\xi)}$

$$K^{(\xi)} = V_\infty^{(\xi)} = \overline{\bigcup_{n=1}^{+\infty} V_n^{(\xi)}},$$

where:

$$V_0^{(\xi)} = \Gamma = \{A, B\}, \quad V_n^{(\xi)} = \bigcup_{i=1}^{4} \psi_i^{(\xi_n)} \left(V_{n-1}^{(\xi)} \right).$$

The set $\Gamma = \{A, B\}$ is defined in previous Section 2 (see (2.3)). For every given ξ, the sequence of sets $V_n^{(\xi)}$, $n \geqslant 0$, is monotone increasing, as a consequence of both the inclusions

$$\Gamma \subset \bigcup_{i=1,\ldots,4} \psi_i^{(1)}(\Gamma) = \Psi^{(1)}(\Gamma), \qquad \Gamma \subset \bigcup_{i=1,\ldots,4} \psi_i^{(2)}(\Gamma) = \Psi^{(2)}(\Gamma).$$

Moreover each set $V_n^{(\xi)}$ is a discrete graph, two neighboring points $p \sim_n q$ in $V_n^{(\xi)}$ being any pair (p,q) belonging to the same set $\psi_{i|n}^{(\xi)}(\Gamma)$ where

$$\psi_{i|n}^{(\xi)}(\mathcal{O}) = \psi_{i_1}^{(\xi_1)} \circ \cdots \circ \psi_{i_n}^{(\xi_n)}(\mathcal{O}) \tag{3.2}$$

for some finite sequence $i|n = (i_1, \ldots, i_n), i_1, \ldots, i_n \in \{1,2,3,4\}$ and any set $\mathcal{O} \subset \mathbb{R}^2$.

On the set $K^{(\xi)}$ an energy form $\mathcal{E}^{(\xi)}[u]$ is defined as limit of an increasing sequence of quadratic forms constructed by finite difference schemes

$$\mathcal{E}^{(\xi)}[u] = \lim_{n \to +\infty} \mathcal{E}_n^{(\xi)}[u] \tag{3.3}$$

where

$$\mathcal{E}_n^{(\xi)}[u] = \rho^n/2 \sum_{p \in V_n^{(\xi)}} \sum_{q \sim_n p} (u(p) - u(q))^2, \tag{3.4}$$

and $\rho = 4$. The set $K^{(\xi)}$, as well as, the energy form $E^{(\xi)}[u]$ depend on the specific sequence ξ, which guided their construction. We now consider this dependence.

Let $h_j^{(\xi)}(n)$ denote the frequency of the occurrence of the family j in the finite sequence $\xi|_n$, $n \geqslant 1$:

$$h_j^{(\xi)}(n) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{\xi_i = j\}}, \; j = 1, 2.$$

The term $h_j^{(\xi)}(n)$ gives the frequency by which each the map $\Psi^{(j)}$ occurs, up to the step n, in our construction of the graph $V_n^{(\xi)}$ and, eventually as $n \to +\infty$, the frequency by which each the map $\Psi^{(j)}$ occurs in the construction of the set $K^{(\xi)}$.

Suppose that there exists an asymptotic frequency of occurrence, p_j such that:

$$h_j^{(\xi)}(n) \to p_j, \; n \to +\infty, \tag{3.5}$$

where $0 \leq p_j \leq 1,\ p_1 + p_2 = 1$ and the rate of the convergence is given by

$$|h_j^{(\xi)}(n) - p_j| \leq \frac{c*}{n},\ j = 1, 2,\ (n \geq 1). \tag{3.6}$$

Set

$$d = d^{(\xi)} = \frac{\lg 4}{p_1 \lg \alpha_1 + p_2 \lg \alpha_2}. \tag{3.7}$$

The bilinear form associated with the energy $\mathcal{E}^{(\xi)}$, defined in (3.3), is a regular Dirichlet form on $L^2(K^{(\xi)}, \mathcal{H}^d)$ with (dense) domain $D[\mathcal{E}]$ (see [25], [26] and also [7]).

It can be proved (see [26]) that the set $K^{(\xi)}$ is a d-set (in the sense of [12]) with respect to the Hausdorff measure $\mathcal{H}^d$ where d is as in (3.7).

Moreover any function v in the domain $D[\mathcal{E}]$ of the energy form is Hölder continuous with Hölder exponent

$$\delta = \delta^{(\xi)} = \mathrm{d}^{(\xi)}/2 = \frac{\lg 2}{p_1 \lg \alpha_1 + p_2 \lg \alpha_2} \tag{3.8}$$

and the Harnack inequality for positive "harmonic" functions holds (see [33] Theorem 0.4, [27] and also [26]). The subspace of $D[\mathcal{E}]$ of all functions $u \in D[\mathcal{E}]$, that vanish at the end-points A and B of $K^{(\xi)}$, will be denoted by $D_0[\mathcal{E}]$. The space $D_0[\mathcal{E}]$ turns out to be an Hilbert space with respect to the "intrinsic" norm

$$||u||_{D_0[\mathcal{E}]} = \mathcal{E}^{(\xi)}[u]^{1/2}.$$

In the following, we consider the form $\mathcal{E}$ always on its domain $D_0[\mathcal{E}]$.

Let Ω denote the polygonal domain with vertices A, B, C, D:
$C = (1/2, 1/2 \cdot \tan\theta_1)$ and $D = (1/2, -1/2 \cdot \tan\theta_1)$, where θ_1 is the rotation angle in the similarities of the family $\Psi^{(1)}$ (see Section 1). Note that as $2 < \alpha_1 < \alpha_2 < 4$ then the amplitudes of the two angles θ_1 and θ_2 corresponding to the two (fixed) values of the contraction factors α_1 and α_2 satisfy the relation $0 < \theta_2 < \theta_1 < \pi/2$.

Let K_0 be the segment of end-points A and B. For every $0 < \varepsilon \leq \varepsilon_0 \leq c_1/2$, where $c_1 = 2\tan(\beta^*/4)$ and $\beta^* = \min\{\pi - 2\theta_1, \theta_1\}$, the "$\varepsilon$-neighborhood" of K_0, denoted Σ_ε, is the polygon whose vertices are the points A, P_1, P_2, B, P_3, P_4, where

$$P_1 = \left(\frac{\varepsilon}{c_1}, \frac{\varepsilon}{2}\right), \quad P_2 = \left(1 - \frac{\varepsilon}{c_1}, \frac{\varepsilon}{2}\right), \quad P_3 = \left(1 - \frac{\varepsilon}{c_1}, -\frac{\varepsilon}{2}\right), \quad P_4 = \left(\frac{\varepsilon}{c_1}, -\frac{\varepsilon}{2}\right).$$

We then subdivide Σ_ε as the union of the rectangle $\mathcal{R}_\varepsilon$ and the two triangles $\mathcal{T}_{h,\varepsilon}$, $h = 1, 2$. Here, $\mathcal{R}_\varepsilon$ is the rectangle with vertices P_1, P_2, P_3, P_4; $\mathcal{T}_{1,\varepsilon}$ is the triangle with vertices A, P_1, P_4 and $\mathcal{T}_{2,\varepsilon}$ is the triangle with vertices P_2, B, P_3 (see Figure 2).

For every integer n, let $K_n^{(\xi)}$ be the polygonal curve

$$K_n^{(\xi)} = \bigcup_{i|n} \psi_{i|n}^{(\xi)}(K_0) = \bigcup_{i|n} K_{i|n}^{(\xi)} = \ = \bigcup_{i=1}^{4} \psi_i^{(\xi_n)}\left(K_{n-1}^{(\xi)}\right)$$

for every n and ε as above, we define the "ε-neighborhood", $\Sigma^{(\xi)}_{n,\varepsilon}$, of $K^{(\xi)}_n$ to be the (open) set

$$\Sigma^{(\xi)}_{n,\varepsilon} = \bigcup_{i|n} \Sigma^{(\xi)}_{i|n,\varepsilon},$$

where $\Sigma^{(\xi)}_{i|n,\varepsilon} = \psi^{(\xi)}_{i|n}(\Sigma_\varepsilon)$.

Note that $\Sigma^{(\xi)}_{i|n,\varepsilon}$ is a topological neighborhood of $K^{(\xi)}_n \setminus V^{(\xi)}_n$.

In the domain Ω, taken together with the embedded layer $\Sigma^{(\xi)}_{n,\varepsilon}$ for given n and ε, we now define a *weight*, $w^{(\xi)}_{n,\varepsilon}$, as follows. Let $P = (\zeta, \eta)$ – for some $i|n$ – belong to the set $\Sigma^{(\xi)}_{i|n,\varepsilon}$ then $(\zeta, \eta) = \psi^{(\xi)}_{i|n}(x, y)$ with $(x, y) \in \Sigma_\varepsilon$ then we set

$$w^{(\xi)}_{n,\varepsilon}(\zeta, \eta) = (\alpha^*)^n \ell_\varepsilon^{-1}(x) \tag{3.9}$$

where

$$\ell_\varepsilon(x) = \begin{cases} \varepsilon & \frac{\varepsilon}{c_1} < x < 1 - \frac{\varepsilon}{c_1} \\ \frac{2(c_1 - c_1 x)}{(2+c_1^2)} & 1 - \frac{\varepsilon}{c_1} < x < 1 \\ \frac{2c_1 x}{(2+c_1^2)} & 0 < x < \frac{\varepsilon}{c_1}\,, \end{cases} \tag{3.10}$$

and

$$\alpha^* = \alpha_1^{p_1} \cdot \alpha_2^{p_2}. \tag{3.11}$$

Moreover, we set

$$w_{n,\varepsilon}(\zeta, \eta) = 1 \qquad \text{if} \quad (\zeta, \eta) \notin \Sigma^{(\xi)}_{n,\varepsilon}\ . \tag{3.12}$$

Associated with the weight $w^{(\xi)}_{n,\varepsilon}$, are the Sobolev spaces

$$H^1(\Omega; w^{(\xi)}_{n,\varepsilon}) = \left\{ u \in L^2(\Omega) : \int_\Omega |\nabla u|^2 w^{(\xi)}_{n,\varepsilon} d\zeta d\eta < +\infty \right\} \tag{3.13}$$

and $H^1_0(\Omega; w^{(\xi)}_{n,\varepsilon})$, the latter being the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{H^1(\Omega; w^{(\xi)}_{n,\varepsilon})} = \left\{ \int_\Omega |u|^2 d\zeta d\eta + \int_\Omega |\nabla u|^2 w^{(\xi)}_{n,\varepsilon} d\zeta d\eta \right\}^{\frac{1}{2}}\,,$$

and the "weighted" energy functionals in $L^2(\Omega)$

$$F^{(\xi)}_{n,\varepsilon}([u]) = \begin{cases} \int_\Omega a^{(\xi)}_{n,\varepsilon}(\zeta, \eta) |\nabla u|^2 d\zeta d\eta & \text{if} \quad u \in H^1_0(\Omega; w^{(\xi)}_{n,\varepsilon}) \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus H^1_0(\Omega; w^{(\xi)}_{n,\varepsilon}) \end{cases}\,, \tag{3.14}$$

where the unbounded conductivity coefficient is

$$a^{(\xi)}_{n,\varepsilon}(\zeta, \eta) = \begin{cases} \sigma_n\, w^{(\xi)}_{n,\varepsilon}(\zeta, \eta) & \text{if} \quad (\zeta, \eta) \in \Sigma^{(\xi)}_{n,\varepsilon} \\ 1 & \text{if} \quad (\zeta, \eta) \notin \Sigma^{(\xi)}_{n,\varepsilon} \end{cases} \tag{3.15}$$

with σ_n some given positive constant.

From now on we shall suppose that the contraction factors are "rational", *i.e.*, $\alpha_1, \alpha_2 \in \mathbb{Q}$.

The result is the following with notation as given above:

Theorem 3.1. *Let* $\varepsilon = \varepsilon(n)$ *be an arbitrary sequence such that* $\varepsilon(n) \to 0$ *as* $n \to +\infty$ *and let* $\sigma_n = (\frac{\rho}{\alpha^*})^n$ *for every* n*, where* $\rho = 4$ *and* α^* *is defined in* (3.11)*. Then the sequence of functionals* $F^{(\xi)}_{n,\varepsilon(n)}$*, defined in* (3.14)*,* M*-converges to the following functional* F *as* $n \to +\infty$*:*

$$F[u] = \begin{cases} \int_\Omega |\nabla u|^2 d\zeta d\eta + \mathcal{E}[u|_{K^{(\xi)}}] & \text{if} \quad u \in D_0[E] \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus D_0[E] \end{cases},$$

where $D_0[E] = \{v \in H^1_0(\Omega) : v|_{K^{(\xi)}} \in D_0[\mathcal{E}]\}$*. Here* $v|_{K^{(\xi)}}$ *denotes the trace of* v *on* $K^{(\xi)}$*.*

For the proof of Theorem 3.1, comments and details I refer to [30].

4. Sierpiński gasket

Let Ω be the triangle with vertices D, E, F: $D = (1/2, -\sqrt{3}/2)$, $E = (3/2, \sqrt{3}/2)$ and $F = (-1/2, \sqrt{3}/2)$. Consider the triangle with vertices in the middle-points A, B, C of the sides of Ω:

$$A = (0,0), \ B = (1,0), \ C = (1/2, \sqrt{3}/2)\,.$$

In this triangle the Sierpiński curve can be constructed by iteration of the 3 contractive similarities ψ_1, ψ_2, ψ_3 in $\mathbb{R}^2$:

$$\psi_1(z) = \frac{z}{2}, \quad \psi_2(z) = \frac{z}{2} + \frac{1}{2}, \quad \psi_3(z) = \frac{z}{2} + \frac{1}{4} + i\frac{\sqrt{3}}{4}$$

where $z = x + iy$.

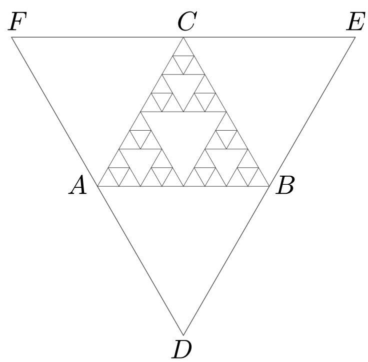

Figure 4. Geometry of the domain

For each integer $n > 0$, consider arbitrary n-tuples of indices $i|n = (i_1, i_2, \ldots, i_n) \in \{1,2,3\}^n$ and define $\psi_{i|n} = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_n}$ and, for every set $\mathcal{O}(\subseteq \mathbb{R}^2)$, $\mathcal{O}^{i|n} = \psi_{i|n}(\mathcal{O})$.

Let $V_0 = \{A, B, C\}$. For every integer $n > 0$ we put

$$V^n = \bigcup_{i|n} V_0^{i|n},$$

where $V_0^{i|n} = \psi_{i|n}(V_0)$, and then

$$V^\infty = \bigcup_{n=1}^{+\infty} V^n .$$

The fractal set $\mathcal{G}$ is obtained by taking the closure $\mathcal{G} = \overline{V}^\infty$ of the set V^∞ in $\mathbb{R}^2$. The set $\partial\mathcal{G} = \{A, B, C\}$ is the (intrinsic) boundary of $\mathcal{G}$. The fractal set $\mathcal{G}$ has Hausdorff dimension $d = \ln 3/\ln 2$ (see Hutchinson [10]).

An energy form $\mathcal{E}[u]$ is also defined on $\mathcal{G}$, which is the limit of an increasing sequence of quadratic forms constructed by finite difference schemes. Namely,

$$\begin{cases} \mathcal{E}[u] = \lim_{n\to+\infty} \mathcal{E}^{(n)}[u] \\ \mathcal{E}^{(n)}[u|_{V^n}] = \rho^n \sum_{i|n}\{(u(\psi_{i|n}(A)) - u(\psi_{i|n}(B)))^2 + \\ (u(\psi_{i|n}(A)) - u(\psi_{i|n}(C)))^2 + (u(\psi_{i|n}(C)) - u(\psi_{i|n}(B)))^2\}, \end{cases} \tag{4.1}$$

where $\rho = 5/3$.

The associated bilinear form $\mathcal{E}(\cdot,\cdot)$ is a regular Dirichlet form on $L^2(\mathcal{G}, \mathcal{H}^d)$, with a domain $D[\mathcal{E}]$ dense in $L^2(\mathcal{G}, \mathcal{H}^d)$. The functions $u \in D[\mathcal{E}]$ turn out to be continuous functions on $\mathcal{G}$, which are indeed Hölder continuous with exponent $\delta = \ln(5/3)/\ln 4$. For the Hölder estimates we refer to Kozlov [14] (see also [25], where Kozlov's result is interpreted as an intrinsic Morrey's imbedding). The subspace of $D[\mathcal{E}]$ of all functions $u \in D[\mathcal{E}]$, that vanish on $\partial\mathcal{G}$, that is, at the points A, B and C of $\mathcal{G}$ will be denoted by $D_0[\mathcal{E}]$. In the following, we consider the form $\mathcal{E}$ always on its domain $D_0[\mathcal{E}]$.

By K_l, $l = 0, 1, 2$ we denote the segments with end-points A and B, B and C, C and A, respectively. We now consider K_0. For every $0 < \varepsilon \le c_1/2$, where $c_1 = 2\tan(\pi/12)$, we define the "ε-neighborhood" of K_0, denoted $\Sigma_{0,\varepsilon}$, to be the polygon whose vertices are the points A, P_1, P_2, B, P_3, P_4, where

$$P_1 = \left(\frac{\varepsilon}{c_1}, \frac{\varepsilon}{2}\right), \quad P_2 = \left(1 - \frac{\varepsilon}{c_1}, \frac{\varepsilon}{2}\right), \quad P_3 = \left(1 - \frac{\varepsilon}{c_1}, -\frac{\varepsilon}{2}\right), \quad P_4 = \left(\frac{\varepsilon}{c_1}, -\frac{\varepsilon}{2}\right).$$

We then subdivide $\Sigma_{0,\varepsilon}$ as the union of the rectangle $\mathcal{R}_{0,\varepsilon}$ and the two triangles $\mathcal{T}_{0,j,\varepsilon}$, $j = 1, 2$. Here, $\mathcal{R}_{0,\varepsilon}$ is the rectangle with vertices P_1, P_2, P_3, P_4; $\mathcal{T}_{0,1,\varepsilon}$ is the triangle with vertices A, P_1, P_4 and $\mathcal{T}_{0,2,\varepsilon}$ is the triangle with vertices P_2, B, P_3. Similarly, for $l = 1, 2$, we construct the "ε-neighborhood" $\Sigma_{l,\varepsilon}$ of K_l, as before, and we decompose $\Sigma_{l,\varepsilon}$, in the union of the rectangle $\mathcal{R}_{l,\varepsilon}$ and the two triangles $\mathcal{T}_{l,j,\varepsilon}$, $j = 1, 2$.

We define the set

$$G_0 = \bigcup_{l=0,1,2} K_l$$

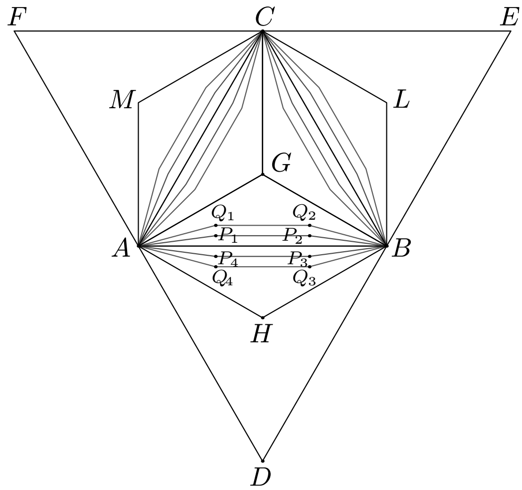

FIGURE 5. Geometry of the layers

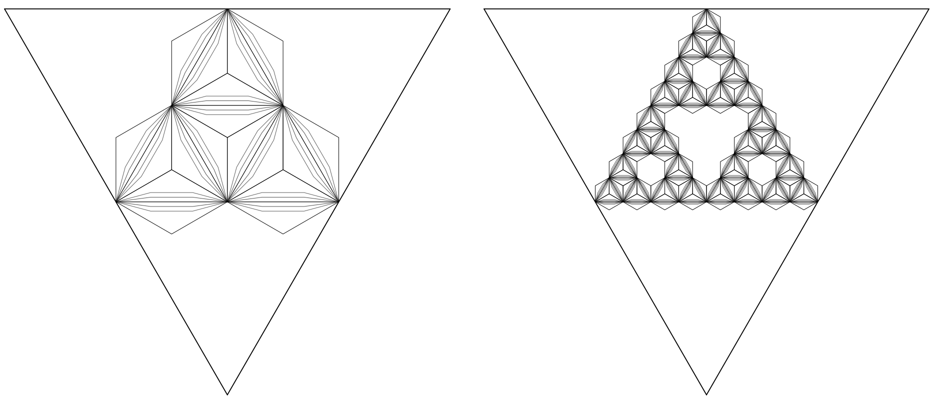

FIGURE 6. “Iterated” layers

and the “ε-neighborhood”

$$\Sigma_\varepsilon = \bigcup_{l=0,1,2} \Sigma_{l,\varepsilon}$$

of G_0. For every n, we consider the (ramified) polygonal curve

$$G^n = \bigcup_{i|n} G_0^{i|n}$$

where $G_0^{i|n} = \psi_{i|n}(G_0)$. Moreover, for every n and ε as above, we consider the "ε-neighborhood"

$$\Sigma_\varepsilon^n = \bigcup_{i|n} \Sigma_\varepsilon^{i|n}$$

of G^n, where $\Sigma_\varepsilon^{i|n} = \psi_{i|n}(\Sigma_\varepsilon)$, see Figure 6. Note that Σ_ε^n is a topological (open) neighborhood of $G^n \setminus V^n$.

In the domain Ω, taken together with the embedded layer Σ_ε^n, for given n and ε we define the *weight*, w_ε^n as follows. Let P – for some l and $i|n$ – belong to the boundary $\partial\Sigma_{l,\varepsilon}^{i|n}$ of $\Sigma_{l,\varepsilon}^{i|n}$. Let $P^\perp$ be the orthogonal projection of P on $K_l^{i|n}$. If (x,y) belongs to the segment with end-points P and $P^\perp$, we set

$$w_\varepsilon^n(x,y) = \begin{cases} \frac{2+c_1^2}{4|P-P^\perp|} & \text{if} \quad (x,y) \in \overset{\circ}{\mathcal{T}}{}_{l,j,\varepsilon}^{i|n} \quad \text{and } j=1,2 \\ \frac{1}{2|P-P^\perp|} & \text{if} \quad (\xi,\eta) \in \mathcal{R}_{l,\varepsilon}^{i|n} \end{cases} \tag{4.2}$$

where $|P - P^\perp|$ is the (Euclidean) distance between P and $P^\perp$ in $\mathbb{R}^2$, $\mathcal{T}_{l,j,\varepsilon}^{i|n} = \psi_{i|n}(\mathcal{T}_{l,j,\varepsilon})$, $\mathcal{R}_{l,\varepsilon}^{i|n} = \psi_{i|n}(\mathcal{R}_{l,\varepsilon})$. Then, we set

$$w_\varepsilon^n(x,y) = \begin{cases} \sigma_n w_\varepsilon^n(\xi,\eta) & \text{if} \quad (\xi,\eta) \in \Sigma_\varepsilon^n \\ 1 & \text{if} \quad (\xi,\eta) \in \Omega \setminus \Sigma_\varepsilon^n\,, \end{cases} \tag{4.3}$$

where $\sigma_n > 0$ are constants that will be specified later. We note that, for fixed n and ε, the weight w_ε^n belongs to the Muckenhoupt class A_2.

For given n and ϵ, the weighted Sobolev spaces $H^1(\Omega; w_\varepsilon^n)$ and $H_0^1(\Omega; w_\varepsilon^n)$ are defined as in previous Section 2 (see (2.9)). We now consider the (quadratic) functional F_ϵ^n in the Hilbert space $L^2(\Omega)$ with extended real values

$$F_\varepsilon^n[u] = \begin{cases} \int_\Omega a_\varepsilon^n(x,y)|\nabla u|^2 dxdy & \text{if} \quad u \in H_0^1(\Omega; w_n^\varepsilon) \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus H_0^1(\Omega; w_n^\varepsilon) \end{cases} \tag{4.4}$$

and the quadratic functional F defined in the Hilbert space $L^2(\Omega)$ with extended real values:

$$F[u] = \begin{cases} \int_\Omega |\nabla u|^2 dxdy + \mathcal{E}[u|_\mathcal{G}] & \text{if} \quad u \in D_0[E] \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus\ D_0[E] \end{cases} \tag{4.5}$$

where $D_0[E] = \{u : \ u \in H_0^1(\Omega),\ u|_\mathcal{G} \in D_0[\mathcal{E}]\}$.

Let me note, incidentally, that the associated bilinear form

$$E(u,v) = \int_\Omega \nabla u \nabla v dxdy + \mathcal{E}(u|_\mathcal{G}, v|_\mathcal{G}) \tag{4.6}$$

with domain $D_0[E]$ is a regular Dirichlet form in $L^2(\Omega)$, in particular the domain $D_0[\mathcal{E}]$ is dense in $L^2(\Omega)$, see, e.g., [11].

We can now state the convergence result for the Sierpiński gasket, with notation as given above:

Theorem 4.1. *Let $N = 3$, $\alpha = 2$, $\rho = 5/3$. Let $\varepsilon_n = (\rho/N)^n \omega_n$, where $\omega_n \le c_1/2$ is an arbitrary sequence such that $\omega_n \to 0$ as $n \to +\infty$, and let $\sigma_n = (\rho/\alpha)^n$. Then the sequence of functionals $F^n_{\varepsilon_n}$ defined in (4.4) M-converges to the functional F in $L^2(\Omega)$ defined in (4.5), as $n \to +\infty$.*

For the proof of Theorem 4.1, comments and details I refer to [29].

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On the Double Layer Potential

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Dedicated to Vladimir G. Maz'ya on the occasion of his 70th birthday

Abstract. Neumann's classical integral equation with the double layer potential operator is considered on different spaces of boundary charges, such as continuous data, $L^2(\Gamma)$ and energy trace spaces. Corresponding known results for different classes of boundaries are collected and discussed in view of their consequences for collocation or Galerkin boundary element methods.

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1. Introduction

The use of boundary potentials for solving elliptic boundary value problems has a long history going back to Green and Gauss, and results in the reduction of the problem to boundary integral equations. This program (see [26]) originated a vast development of mathematics; Vladimir Maz'ya gives a comprehensive survey in [38]. Employing the double layer potential, Dirichlet's problem for the Laplacian leads to Carl Neumann's celebrated integral equation of the second kind which can be solved explicitly with Neumann's successive iteration. Although the integral equations of the first kind are closer related to variational principles [25, 50], the equations of the second kind are mostly preferred, not only in theory but also for their numerical solution [2]. During the 60s of the last century, the panel method based on boundary element collocation for Neumann's equation was used by Hess and Smith [23] in aircraft industry. At about the same time, Vladimir Maz'ya and Josef Kral with their colleagues accomplished the significant extension of Radon's theory from two to three dimensions [5, 6, 7, 28, 29, 30, 31]. For some time the numerical boundary element method suffered since their system matrices are fully populated; then Rokhlin [69] with his multipole approximation started the development of modern efficient and fast methods with boundary integral equations [58, 68, 70]. But reliability and accuracy of these simulation techniques rely on rigorous mathematical analysis. In this paper I collect properties of the double layer potential with a particular view on its relations to collocation and Galerkin boundary element methods for pointing out some still existing gaps between rigorous, known results and the needs for numerical analysis. For panel methods, i.e., collocation methods with continuous boundary data on general polyhedrons, and for classical Galerkin methods with L^2 data on Lipschitz boundaries stability and convergence is still not proved rigorously, and for the realization of Neumann's method on Lipschitz boundaries in energy spaces due to Costabel [10] and [72] we need to use Galerkin–Petrov methods with different trial and test functions.

2. The Laplacian

Carl Friedrich Gauss proposed in [17] for the solution of the Dirichlet problem

$$\Delta u = 0 \ \text{ in } \Omega \subset \mathbb{R}^n\,,\ n = 2 \ \text{ or } 3\,;\ u|_\Gamma = \varphi \ \text{ on } \ \Gamma = \partial\Omega$$

where the boundary values φ are given, the use of the double layer potential

$$u(x) = -\tfrac{2^{1-n}}{\pi} \int\limits_\Gamma \mu(y)\partial_{n_y} E(x,y) = \tfrac{2^{1-n}}{\pi} \int\limits_{y\in\Gamma} \mu d\Omega_x(y) =: W\mu(x) \qquad (2.1)$$

for $x \in \Omega$ where

$$E(x,y) = \begin{cases} -\ln|x-y| & \text{for } n = 2, \\ |x-y|^{-1} & \text{for } n = 3 \end{cases}$$

is the fundamental solution of the Laplacian and

$$d\Omega_x(y) = \frac{n(y)\cdot(y-x)}{|y-x|^n} do(y) \qquad (2.2)$$

denotes the solid angle measure with do the surface measure on Γ. By $n(y)$ we denote the exterior unit normal vector on Γ. If we assume that the sectorial limits of the double layer potential $u(x)$ in (2.1) for $x \to x_0 \in \Gamma$ exist (perhaps only almost everywhere) then one obtains for the yet unknown charge density $\mu \in \mathfrak{X}(\Gamma)$ the boundary integral equation

$$\mu = L\mu + \varphi := \Big(\tfrac{1}{2}I + K\Big)\mu + \varphi \qquad \text{on } \Gamma\,, \qquad (2.3)$$

where $\mathfrak{X}(\Gamma)$ denotes an appropriate space of charges. In the classical work by Carl Friedrich Gauss [17] and then by Carl Neumann [55, 56, 57] is $\mathfrak{X}(\Gamma) = C^0(\Gamma)$ the Banach space of continuous functions on Γ. The linear operator L in (2.3) is given by

$$L\mu(x) = \tfrac{1}{2}\mu(x) + K\mu(x) = -\tfrac{2^{1-n}}{\pi} \int\limits_{y\in\Gamma\setminus\{x\}} \big(\mu(y) - \mu(x)\big) d\Omega_x(y)\,, \qquad (2.4)$$

and

$$K\mu(x) = -\tfrac{2^{1-n}}{\pi} \int\limits_{\Gamma\setminus\{x\}} \mu d\Omega_x + \Big(\tfrac{2^{1-n}}{\pi}\Omega_x - \tfrac{1}{2}\Big)\mu(x) \ \text{ with } \Omega_x = \int\limits_{\Gamma\setminus\{x\}} d\Omega_x \qquad (2.5)$$

is the double layer boundary integral operator. Note that it contains a contribution of the Dirac functional due to corners and edges of Γ.

As the second fundamental problem we consider the Neumann problem where for u with $\Delta u = 0$ in Ω, the Neumann data are given:

$$\partial_n u|_\Gamma = \nabla u \cdot n do|_\Gamma = d\Psi \ (= \psi do) \text{ on } \Gamma \text{ satisfying } \int_\Gamma d\Psi = 0\,. \tag{2.6}$$

Already Gauss used positive measures as charges for the simple layer potential, and Plemelj [59, p. 52] introduced the notion of boundary flux for the given normal derivative defined by

$$\langle v, \Psi \rangle = \int_\Gamma v d\Psi = \int_\Gamma v \partial_n u \tag{2.7}$$

for all continuous test functions v; which motivated Radon to develop the theory of signed measures based on functions of bounded variation characterizing the bounded linear functionals on $\mathfrak{X} = C^0(\Gamma)$ [61, 62, 63]. The solution of the Neumann problem (2.6) is now based on the simple layer potential

$$u(x) = \int_{y\in\Gamma} E(x,y) d\mathsf{P} \ \Big(= \int_{y\in\Gamma} E(x,y)\varrho(y)do(y)\Big) \tag{2.8}$$

where the Radon measure P is to be determined from the boundary integral equation

$$\mathsf{P} = \Big(\tfrac{1}{2}I - K^*\Big)\mathsf{P} + \Psi \text{ on } \Gamma \text{ for } \mathsf{P} \in \mathfrak{X}_0^* := \{\mathsf{P} \in \mathfrak{X}^* \text{ with } \int_\Gamma d\mathsf{P} = 0\}\,. \tag{2.9}$$

If K in (2.4) is considered as a continuous linear operator on the Banach space $\mathfrak{X} = C^0(\Gamma)$ equipped with the supremum norm, then K^* in (2.9) is the adjoint to K defined on $\mathfrak{X}^*(\Gamma) = BV(\Gamma)$, the linear space of Radon measures P equipped with the norm $\int_\Gamma |d\mathsf{P}|$ of total variation.

However, the potential theory and boundary integral equations can also be considered for other function spaces on Γ such as $\mathfrak{X} = L^p(\Gamma)$ or $\mathfrak{X} = H^\sigma(\Gamma)$, the Sobolev–Slobodetskij trace spaces on Γ and their dual spaces $\mathfrak{X}^*$. Hence, it is natural to ask:

1. Which are reasonable assumptions on $\Gamma = \partial\Omega$?
2. Which function space $\mathfrak{X}$ on Γ is a suitable choice?
   Can Fredholm theory be applied to the boundary integral equations (2.3) and (2.9) and what can be said about the corresponding spectra?
   Does Neumann's series for the integral equations converge and provide an efficient solution method?
3. Can the discretization of these equations lead to effective and simple numerical solution algorithms for solving the fundamental Dirichlet or Neumann problems?

In fact, all these three topics are closely related to each other. It was Radon in [63] who introduced the concept of the Fredholm radius of a linear continuous operator

$A : \mathfrak{X} \to \mathfrak{X}$ in the Banach space $\mathfrak{X}$ with norm $\|v\|_\wp$. Let us consider the equation

$$\lambda v = Av + f \text{ in the Banach space } \mathfrak{X} \tag{2.10}$$

with given $f \in \mathfrak{X}$ and $\lambda \in \mathbb{C}$ a parameter. Then the essential norm of A is defined as

$$\begin{aligned} &\|A\|_{\wp \text{ ess}} := \inf_C \|A - C\|_\wp, \quad \text{where } C \text{ ranges} \\ &\text{over all completely continuous operators on } \mathfrak{X}\,. \end{aligned} \tag{2.11}$$

Then the classical Fredholm theory holds for (2.10) if

$$|\lambda| > \|A\|_{\wp \text{ ess}} = (\text{Fredholm radius})^{-1}\,. \tag{2.12}$$

If $\lambda = 1$ then for the convergence of the Neumann series

$$v^{m+1} = Av^m + f\,,\ m \in \mathbb{N}_0 \tag{2.13}$$

in $\mathfrak{X}$, a sufficient condition is that the spectral radius defined as

$$r_\wp(A) := \lim_{n\to\infty} (\|A^n\|_\wp)^{1/n} \text{ satisfies } r_\wp(A) < 1\,. \tag{2.14}$$

3. The two-dimensional case

Let $\Omega \subset \mathbb{R}^2$ with boundary Γ be a simple, closed Jordon curve which at least is rectifiable.

3.1. Continuous boundary data

For strictly convex Ω, Neumann showed in his work [55, 56, 57] the following theorem. (See also [35] by Kral and Netuka.)

Theorem 3.1. *Let Ω be convex and not one of the exceptional domains in* [35]*; $\mathfrak{X} = C^0(\Gamma)$ equipped with the supremum norm and define the oscillation of $v \in C^0(\Gamma)$ by*

$$\text{osc}(v) := \sup_{x,y\in\Gamma} |v(x) - v(y)|\,. \tag{3.1}$$

Then L is a contraction with respect to the oscillation:

$$\text{osc}\left(\frac{1}{2\pi}\int_{\Gamma\setminus\{x\}} |d\Omega_x|\right) \le \delta < \frac{1}{2}\,,\ \text{osc}(K\mu) \le \delta\,\text{osc}\,\mu \text{ and } \text{osc}(L\mu) \le \left(\frac{1}{2} + \delta\right)\text{osc}\,\mu\,. \tag{3.2}$$

If we replace the supremum norm on $C^0(\Gamma)$ by the equivalent norm [27],

$$\|v\|_\wp := \text{osc}(u) + \beta \sup |u(x)| \text{ with } 0 < \beta < 1 - \delta\,, \tag{3.3}$$

then

$$\|L\|_\wp \le q < 1 \text{ where } q = (1 + \beta + \delta)/(2 + \beta) < 1\,, \tag{3.4}$$

and (2.13) *with $A = L$ converges in $C^0(\Gamma)$.*

If Ω is bounded by a simple, closed Lyapunov curve $\Gamma \in C^{1,\alpha}$, $0 < \alpha \le 1$ which is not necessarily convex, then the following results are due to Poincaré [60], Plemelj [59] and Radon [62].

Theorem 3.2. *Let* $\Gamma \in C^{1,\alpha}$, $0 < \alpha \le 1$, $\mathfrak{X} = C^0(\Gamma)$ with $\|v\|_\wp = \sup_{x\in\Gamma} |u(x)|$. *Then the Fredholm radius is infinite,*

$$\|K\|_{\wp \text{ ess}} = 0\,.$$

The spectrum of the Neumann operator $-2K$ *is a real, countable point spectrum* $\{\lambda_\ell\}$, $\ell \in \mathbb{N}_0$; $\lambda_0 = 1$ *is a simple eigenvalue and* $|\lambda_\ell| < 1$ *for* $\ell \in \mathbb{N}$. *Moreover, the spectral radius of* L *satisfies*

$$r_\wp(L) = \lim_{n\to\infty} (\|L^n\|_\wp)^{1/n} = q < 1\,. \tag{3.5}$$

Hence, (2.10) *with* $A = L$ *converges in* $C^0(\Gamma)$.

Since the equation (2.9) holds in $\mathfrak{X}^*$, the dual space to $C^0(\Gamma)$, Theorem 3.2 also applies to K^* and L^* in $\mathfrak{X}^* = BV(\Gamma)$, the space of Radon measures on Γ, correspondingly. On the other hand, for $\Gamma \in C^{1,\alpha}$, K^* is also completely continuous on $C^0(\Gamma)$ and the classical Fredholm theorems hold for (2.3) and (2.9) where $\mathsf{P} = \varrho ds$ with $\varrho \in C^0(\Gamma)$.

Radon in [62] reduced the requirement for Γ significantly when defining the curves of bounded rotation, i.e., Γ has the parametrization

$$x(s) = x(s_0) + \int_{s_0}^{s} \big(\cos\vartheta(t)\,,\, \sin\vartheta(t)\big)^\top dt\,;$$

where $\vartheta(s)$ is a function of bounded variation, $\int_\Gamma |d\vartheta| < \infty$. For these curves, the angular jump $\gamma(t) := \vartheta(t+0) - \vartheta(t-0)$ is well defined, and Radon proved:

Theorem 3.3. *If* Γ *is a curve of bounded rotation then the Gauss and Green formulae are valid; the double layer operator satisfies*

$$\|K\|_{\wp \text{ ess}} = \sup \left|\tfrac{\gamma}{2\pi}\right|, \tag{3.6}$$

and $KV = VK^*$ *on* $\mathfrak{X}^*(\Gamma)$ *where* V *is the simple layer operator given by* (2.8) *with* $x \in \Gamma$ *and* $\wp$ *denotes the supremum norm. Hence, if*

$$\sup\left|\tfrac{\gamma}{2\pi}\right| < \tfrac{1}{2} \ \text{ and } |\lambda| > \sup\left|\tfrac{\gamma}{\pi}\right|, \tag{3.7}$$

the Fredholm theorems are applicable for (2.3) *and* (2.9), *and for these* λ, *the spectrum of* $-2K$ *has the same properties as in Theorem* 3.2.

Moreover, under condition (3.7), *there holds* $\|L\|_{\wp \text{ ess}} < 1$ *and, in addition,* (3.5) *remains valid.*

Now (2.9) must be considered in $\mathfrak{X}^* = \big(C^0(\Gamma)\big)^* = BV(\Gamma)$ since K^* is on $C^0(\Gamma)$ not bounded anymore due to Netuka [52].

The final result on the additional requirements on a rectifiable boundary Γ for Fredholm's theorems to hold for (2.3) in $C^0(\Gamma)$ and (2.9) in $\big(C^0(\Gamma)\big)^*$ is due to Kral in [32]: The condition

$$\lim_{r\to 0} \sup_{x\in\Gamma} \frac{1}{2\pi} \int_{0<|x-y|\le r} |d\Omega_x(y)| < \tfrac{1}{2} \tag{3.8}$$

is necessary and sufficient for an admissible boundary $\Gamma \subset \mathbb{R}^2$.

Because of (3.6) resp. (3.8) and (2.11), the operator $2K$ can be written as

$$2K = Q + C \text{ where } \|Q\|_{\wp} < 1 \text{ and } C \text{ is completely continuous}. \tag{3.9}$$

Hence, $(I - Q)$ is invertible and (2.3), i.e.,

$$\mu - (Q + C)\mu = 2\varphi \tag{3.10}$$

is equivalent to

$$\mu = (I - Q)^{-1}C\mu + 2(I - Q)^{-1}\varphi\,, \tag{3.11}$$

a Fredholm integral equation of the second kind. Since (3.11) has this particular structure, corresponding boundary element approximations with piecewise linear continuous splines $\mu_h \in \mathcal{S}_h$ on a quasi regular family of grids where h denotes the maximal meshwidth, and point collocation at the grid points converges to μ in $C^0(\Gamma)$ if h tends to zero [4]. Also piecewise constant approximation and point collocation at the element midpoints converges (see the arguments in [27] for the two-dimensional case). Moreover, Neumann's iteration (2.10) converges also if A is replaced by the approximating operators L_h defined by point collocation on $\mathcal{S}_h$ provided h is small enough.

3.2. Square integrable boundary data

The boundary integral equation (2.3) on the space $\mathfrak{X} = L^2(\Gamma)$ for a Lipschitz boundary Γ was analyzed by Verchota [73] who showed that $K : L^2(\Gamma) \to L^2(\Gamma)$ is continuous and $(\frac{1}{2}I - K)$ is invertible. Hence, now in (2.8) we can use $\varrho \in L^2(\Gamma)$, the adjoint operator K^* is continuous as well and it can be shown that $(\frac{1}{2}I + K^*)$ is invertible on $L^2_0(\Gamma) = \mathfrak{X}^*_0 = \{\varrho \in L^2(\Gamma) \text{ with } \int\limits_\Gamma \varrho do = 0\}$. For Lipschitz Γ, however, it is not known whether the essential norm of $2K$ in $L^2(\Gamma)$ is less than one.

If Γ is a convex Lipschitz curve, then Fabes, Sand and Keun Seo proved in [16] that the spectral radius satisfies (3.5) (with $\wp = L^2(\Gamma)$). Hence, in this case (2.10) with $L = A$ converges in $L^2(\Gamma)$.

If Γ is a Lyapunov curve $\Gamma \in C^{1,\alpha}$, $0 < \alpha \leq 1$, then the kernel in (2.2) is weakly singular as $|x - y|^{\alpha-1}$ and then K as well as K^* are compact operators in $L^2(\Gamma)$ (see [47, Theorem 7.3.2]). Moreover, then the spectra in $L^2(\Gamma)$ and in $C^0(\Gamma)$ are identical. Hence, Theorem 3.2 remains valid in $\mathfrak{X} = L^2(\Gamma)$. For approximations with $\mathcal{S}_h \subset L^2(\Gamma)$ and $h \to 0$, Galerkin's method will be asymptotically stable and converges quasi-optimally in $L^2(\Gamma)$ (see, e.g., [24]).

When Γ is piecewise Lyapunov and has finitely many corners on Γ, then Shelepov proved in [71] that $\|K\|_{L^2 \text{ ess}} = \frac{1}{2}\sin\sup\left|\frac{\gamma}{2}\right|$. Hence, Theorem 3.3 remains valid with $\left|\frac{\gamma}{2\pi}\right|$ replaced by $\frac{1}{2}\sin\left|\frac{\gamma}{2}\right|$. If in this case Γ has no cusps nor peaks and if $\sin\sup\left|\frac{\gamma}{2}\right| < 1$ then (2.3) is equivalent to (3.11) but now in $\mathfrak{X} = L^2(\Gamma)$. Consequently, also in this case the boundary element Galerkin methods are asymptotically stable and convergent in $L^2(\Gamma)$ for $h \to 0$. Also Neumann's iteration (2.13) converges in $L^2(\Gamma)$ not only for $A = L$ but also for the Galerkin approximations $A = L_h$. (For the proof of this statement use, e.g., arguments as in [4].)

For Γ piecewise smooth having a finite number of corner points, the singularities of the solution at the corners must be incorporated into the space $\mathfrak{X}$ and its adjoint. This approach was developed by Grachev and Maz'ya in [19, 20, 21]. If Γ has inward or outward peaks this was extended by Maz'ya and Solov'ev in a series of papers (see [42, 43, 44] and the works cited therein), Maz'ya and Poborchii [41] and by Duduchava and Silbermann [12].

If Γ is a Lipschitz boundary then due to Verchota [73] the operators $\frac{1}{2}I_{(}\mp_{)}K$ and $\frac{1}{2}I_{(}\mp_{)}K^*$ are Fredholm and invertible on $L^2(\Gamma)$ (or $L^2_0)(\Gamma)$). However, for proving stability and convergence of Galerkin's method, the additional structure as in (3.9) and (3.11) is not yet known. Hence, in this case stability and quasi optimal convergence can only be proved for the least squares method, applied to (2.3) and (2.9), respectively.

4. The three-dimensional case

In the three-dimensional case let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\Gamma = \partial\Omega$ whose regularity properties will be specified later on; but at least Ω is a Borel set with finite perimeter (see [38, p. 184]). Then $d\Omega_x(y)$ becomes a Borel measure and we assume that its total variation is uniformly bounded,

$$\sup_{x\in\mathbb{R}^3\setminus\Gamma} \int_{y\in\Gamma} |d\Omega_x(y)| < \infty\,. \tag{4.1}$$

4.1. Continuous boundary data

For a convex domain Ω (but not one of the exceptional domains in [35]) Neumann in [55, 56, 57] showed for $C^0(\Gamma) = \mathfrak{X}$ and the oscillation (3.1) the decisive inequality

$$\operatorname{osc}\left(\frac{1}{4\pi}\int_{\Gamma\setminus\{x\}} d\Omega_x\right) \le \delta < \tfrac{1}{2}\,. \tag{4.2}$$

With the norm (3.3) on $C^0(\Gamma)$, also in the three-dimensional case, the operator L becomes a contraction satisfying (3.4). Hence, Neumann's classical iteration (2.13) with $A = L$ converges in $C^0(\Gamma)$. Netuka showed in [54] that in this case the essential norm of K satisfies

$$\|K\|_{\wp\,\mathrm{ess}} = \lim_{\delta\to 0}\,\sup_{x\in\Gamma}\frac{1}{4\pi}\int_{0<|x-y|<\delta} d\Omega_x(y) \ \text{ where } \|u\|_\wp = \sup\nolimits_{x\in\Gamma}|u(x)|\,. \tag{4.3}$$

For a not necessarily convex domain Ω with Γ a Lyapunov surface in $C^{1,\alpha}$, $0<\alpha\le 1$, the kernel $n(y)\cdot(y-x)\,/\,|x-y|^3$ of the solid angle (2.2) is weakly singular [47, Theorem 7.3.2] and K as well as K^* are completely continuous operators on $C^0(\Gamma)$. Moreover, Theorem 3.2 remains valid due to the relation between the eigenvalues λ_ℓ and the Dirichlet integrals D^+_ℓ , D^-_ℓ of simple layer potentials (2.8) in Ω and $\mathbb{R}^3\setminus\overline{\Omega}$, respectively, with charges ϱ_ℓ which are eigensolutions of $-2K^*$,

$$\lambda_\ell = (D^+_\ell - D^-_\ell)\,/\,(D^+_\ell + D^-_\ell)\,. \tag{4.4}$$

Because of (3.5), Neumann's iteration (2.13) converges in $C^0(\Gamma)$. Moreover, the approximate solutions of (2.3) with piecewise linear or piecewise constant boundary elements $\mu_h \in \mathcal{S}_h$ and point collocation converge in $C^0(\Gamma)$ with meshwidth $h \to 0$ (see [2, 3, 27, 74]).

Radon's famous work on measures with bounded variation [61, 63] was developed in combination with his analysis of the logarithmic potential [62], but for three-dimensional potential theory only from the 60's on, Radon's approach was extended by Burago, Maz'ya, Sapozhnikova [5, 6], Burago, Maz'ya [7] and by Král [28, 29, 30, 31]; also Netuka [53]. For a boundary satisfying (4.1) they proved for the Banach space $\mathfrak{X}(\Gamma) = C^0(\Gamma)$ with the supremum norm:

Theorem 4.1. *Condition* (4.1) *is necessary and sufficient for the double layer potential* (2.1) *with* $\mu \in C^0(\Gamma)$ *to have limit values* $\varphi \in C^0(\Gamma)$ *on* Γ *from* Ω *and limit values* $\varphi + 2K\mu$ *from* $\mathbb{R}^3 \setminus \overline{\Omega}$. *The boundary integral operator becomes a bounded linear mapping:* $K : C^0(\Gamma) = \mathfrak{X} \to \mathfrak{X}$. *For the double layer boundary integral operator and the supremum norm* $\wp$ *one has*

$$\|K\|_{\wp \text{ ess}} = \lim_{\delta \to 0} \sup_{x \in \Gamma} \tfrac{1}{4\pi} \int_{0<|x-y|<\delta} |d\Omega_x(y)| \,. \tag{4.5}$$

Now let the essential norm (4.5) satisfy

$$\|K\|_{\wp \text{ ess}} < \tfrac{1}{2} \,. \tag{4.6}$$

In this case, we again have that the relation (3.9) and equation (2.3) is equivalent to (3.10) and (3.11). Moreover, for $|\lambda| > 2\|K\|_{\wp \text{ ess}}$, the spectrum of $-2K$ is a real, countable point spectrum $\{\lambda_\ell\}$, $\ell \in \mathbb{N}_0$; $\lambda_0 = 1$ is a simple eigenvalue and for λ_ℓ and $\ell \in \mathbb{N}$ relation (4.4) is satisfied. As a consequence, for the spectral radius of L there holds (3.5), and (2.10) with $A = L$ converges in $C^0(\Gamma)$, in this three-dimensional case as well.

If for Γ the assumption (4.6) is satisfied then boundary element collocation with piecewise constant elements $\mathcal{S}_h$ and appropriate collocation points, both on a family of quasi-regular triangulations of Γ, is asymptotically stable and converges quasi-optimally for h tending to zero (see [3, 27, 64, 65, 66, 74]).

There are already many surfaces Γ satisfying condition (4.6) such as some piecewise Lyapunov surfaces with a finite number of corners and edges including those already investigated by Carleman [8], and C^1-surfaces whose normal vector's modulus of continuity ω satisfies $\int_0^1 \varrho^{-1}\omega(\varrho)d\varrho < \infty$ (see Maz'ya [38, § 2] and further examples in [5, 6, 7]). However, at non-convex corner points $x \in \Gamma$ as for general polyhedral domains, the property (4.6) can be violated. ($x \in \Gamma$ is called a convex corner point if the cone $C_x := \{w = \lim_{\delta \to 0} (z-x)\frac{t}{\delta}$ where $z \in \overline{\Omega}$, $|x-z| \le \delta$ and $0 \le t \in \mathbb{R}\}$ or its complement $\mathbb{R}^3 \setminus C_x$ is convex.) But the Fredholm theorems might still be valid since due to Gohberg and Marcus [18] there holds:

Lemma 4.1.1. *Let $A : \mathfrak{X} \to \mathfrak{X}$ be a bounded linear operator and $\mathfrak{X}$ a given Banach space. Then*

$$r_{\text{ess}}(A) := \varliminf_{n\to\infty} (\|A^n\|_{\wp\text{ess}})^{\frac{1}{n}} = \inf_{\wp} \|A\|_{\wp\text{ess}} \tag{4.7}$$

$$= \inf\{r \geq 0 \text{ where for } \lambda \in \mathbb{C} \text{ with } |\lambda| > r \text{ the operator } (A - \lambda I) \text{ is Fredholm}\}$$

and where $\wp$ traces the family of all equivalent norms on $\mathfrak{X}$.

For $\mathfrak{X} = C^0(\Gamma)$, $\wp$ in (4.7) denotes all the norms which induce the topology of uniform convergence of functions on Γ. Hence, in order to extend the family of domains for which the asymptotic stability and convergence of boundary element collocation can be proved, we try to find equivalent norms for which we obtain a smaller essential norm than in (4.5). Motivated by the example in [36] let us consider the class of weighted supremum norms associated with the family of weight functions $w \in L^\infty(\Gamma)$ satisfying $0 < c^- \leq w(x) \leq 1$ and equip $C^0(\Gamma)$ for given w with the norm

$$\|\mu\|_{C^0_w} := \sup_{x\in\Gamma} |\mu(x)/w(x)|\,. \tag{4.8}$$

For Baire functions w, Kral and Medková showed in [33, 34] the following.

Theorem 4.2. *The essential norm of L with respect to the weighted supremum norm* (4.8) *satisfies*

$$\|K\|_{C^0_w\text{ess}} = \lim_{\delta\to 0+} \left\{ \sup_{x\in\Gamma} \frac{1}{4\pi} \int\limits_{0<|x-y|\leq\delta} \frac{w(y)}{w(x)} |d\Omega_x(y)| \right\}. \tag{4.9}$$

Hence, if we find a weight function such that

$$\|L\|_{C^0_w\text{ ess}} < \tfrac{1}{2} \tag{4.10}$$

is satisfied then we have (3.9), and (2.3) is equivalent to (3.10) and (3.11); there holds (3.5), and (2.10) with $A = L$ converges uniformly. Therefore, one is interested in finding weight functions $w(x)$ such that (4.10) holds. For the particular class of so-called rectangular domains which are even not Lipschitz, Kral and co-authors constructed such weight functions [1, 36]. In the case of polyhedral domains, Hansen presented in [22] a procedure providing piecewise sectorially constant weight functions $w(x)$ near corner points guaranteeing (4.10) for a much larger class of polyhedrons than those satisfying (4.6) (which, in fact, is (4.10) with $w \equiv 1$). Note that the conditions in (4.6) and (4.10) are local and that edge points are convex corner points where (4.10) holds if sharp edges are excluded. Hence, Hansen's result only is needed at nonconvex corner points violating (4.6).

If Ω is a polyhedron belonging to Hansen's class, then the spectrum of L on $\mathfrak{X} = C^0(\Gamma)$ in the Fredholm region is contained in the spectrum of L on $L^2(\Gamma)$; the latter was analyzed via Mellin techniques by Rathsfeld in [64, 66] who showed for the spectral radius of L:

$$r_{C^0(\Gamma)}(L) \leq r_{L^2(\Gamma)}(L) = q < 1\,. \tag{4.11}$$

Kral's and Hansen's weight function $w(x)$ is piecewise constant on a particular triangulation Γ_{h_0} of the polyhedral boundary Γ. The family of quasi-regular boundary element triangulations Γ_h for the panel method is called compatible with $w(x)$ if every Γ_h is a refinement of Γ_{h_0}. Then for the panel method with piecewise constant compatible trial functions and collocation, asymptotic stability and quasi-optimal convergence was shown in [37, 75]. Rathsfeld's convergence analysis of the panel method in [64, 66] was based on (4.6) but if [64, (0.3)] is replaced by (4.10) then for the compatible panel method the inequality [64, (2.18)] in his proof still holds and asymptotic stability and convergence are valid. He also developed a quadrature method [65] and a two-grid Nystöm method on graded meshes in [67] under assumption (4.6) which probably can be extended to Hansen's case (4.10).

Medkova showed in [45, 46] that the essential norms of K are invariant under a certain class of diffeomorphisms applied (locally) to Γ which implies that (4.10) also holds for this class of domains obtained from the polyhedrons under these mappings.

Although Neumann's method with the boundary integral equations (2.3) for continuous boundary data and its adjoint (2.9) for Borel measures on Γ are solvable and the corresponding panel collocation methods provide a useful tool for their practical solution on a rather large family of domains, it is still not clear, whether Hansen's procedure always provides a weight function with (4.10) for any arbitrary polyhedral domain. Hence, the stability and convergence of the collocation method for piecewise smooth Γ is in part still open.

4.2. Square integrable boundary data

As in Section 3.2, for $\mathfrak{X} = L^2(\Gamma) = \mathfrak{X}^*$ on a Lipschitz surface Γ, the operators K and K^* are continuous and $\frac{1}{2}I_{(\mp)}K$ and $\frac{1}{2}I_{(\mp)}K^*$ are Fredholm and invertible on $L^2(\Gamma)$ $\big($or $L^2_0(\Gamma)\big)$ and also on the Sobolev spaces $H^s(\Gamma)\big($ or $H^s_0(\Gamma)\big)$ as was proved by Dahlberg [11] and Verchota [73](see also [48]). As in the two-dimensional case, (3.5) is valid for a convex Lipschitz surface $\Gamma \subset \mathbb{R}^3$ [16]. For a Lyapunov boundary $\Gamma \in C^{1,\alpha}$ with $0 < \alpha \leq 1$, the kernel in (2.2) is weakly singular as $|x-y|^{\alpha-2}$ and K as well as K^* are compact linear mappings $L^2(\Gamma)$ (see [47, Theorem 7.3.2]). Moreover, we also have (4.4) for the spectrum implying (3.5) also without convexity of Ω. For a C^1-boundary Γ, the operators K and K^* are still compact on $L^2(\Gamma)$ due to Dahlberg [11] and Fabes, Jodeit and Rivière [15]. For a Lipschitz boundary with sufficiently small Lipschitz character, due to the result by I. Mitrea [49], the operator $2K$ can be written as in (3.9) with $\wp$ the $L^2(\Gamma)$-norm; (2.3) corresponds to (3.10) and (3.11). If Ω is polyhedral belonging to Hansen's class, then the spectral radius of L satisfies (4.11) [64, 65]. In all these cases, the boundary element Galerkin methods are asymptotically stable and convergent in $L^2(\Gamma)$ for decreasing h (see [14, 70]). Moreover, Neumann's series (2.13) with $A = L$ converges in $L^2(\Gamma)$.

If Γ has isolated conical points or edges, the Fredholm radius of K in weighted Hölder–Sobolev spaces was given by Grachev and Maz'ya in [20, 21].

Based on the singular behavior of the solution $u(x)$ at corners and edges of piecewise smooth domains as analyzed by Maz'ya and Plamenevskii [39, 40], Elschner developed in [13, 14] a spline Galerkin method for polyhedral domains where the boundary element mesh is graded according to the singularities, and obtained stability and quasi-optimal error estimates.

For a general Lipschitz boundary Γ, however, stability and convergence of Galerkin's method in $L^2(\Gamma)$ is not yet known. However, the solvability of (2.3) and (2.9) in $L^2(\Gamma)$ guarantees stability and quasi-optimal convergence of least squares boundary element approximations.

4.3. Data in boundary energy spaces

As was seen in (4.4), there is a close relationship between the spectrum of K and the energy of associated potentials. Hence, it seems to be quite natural to use on Γ boundary energy spaces instead of $C^0(\Gamma)$ or $L^2(\Gamma)$. Costabel in [10] reveals that the energy of potentials played a fundamental role already in the works of Gauss and Poincaré who developed in [60] the basic ideas for proving convergence of Neumann's method for a much larger class of elliptic boundary value problems than for the Laplacian. Costabel also presents in [10] a proof alternative to the one in [72] and shows that also $2K$ is a contractive operator in a boundary energy space. To be brief, here let us consider just the Laplacian although this approach is as well valid for all selfadjoint strongly elliptic second-order systems in Lipschitz domains. To this end, we need the boundary integral operator K of the double layer (2.1), of K^*, the adjoint to K, the simple layer potentials (2.8) and the boundary flux of the double layer potential; namely

$$V\varrho(x) := \int\limits_{y\in\Gamma\setminus\{x\}} E(x,y)\varrho(y)do(y)\,,\; D\mu(x) := -\partial_{nx} \int\limits_{y\in\Gamma\setminus\{x\}} \big(\partial_{ny}E(x,y)\big)\mu(y)do(y)$$

for almost every $x \in \Gamma$ which are nontangential limits from Ω and from $\mathbb{R}^3 \setminus \overline{\Omega}$. They define continuous linear mappings (see [9])

$$\begin{aligned} V \,:\, H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)\,, \quad K \,:\, H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)\,, \\ K^* \,:\, H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)\,, \, D \,:\, H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)\,. \end{aligned}$$

(In fact, they all can be extended to a larger class of Sobolev spaces on the Lipschitz boundary Γ.) Since the Green identities are valid, Calderon's projector implies the operator identities

$$\begin{aligned} VD &= \tfrac{1}{4}I - K^2\,, & DV &= \tfrac{1}{4}I - K^{*2} \\ KV &= VK^*\,, & DK &= K^*D\,. \end{aligned}$$

Moreover, the operators $V$and D satisfy coerciveness inequalities, i.e., there exist strictly positive constants c_V and c_D, such that

$$c_V\|\varrho\|^2_{H^{-\frac{1}{2}}(\Gamma)} \le (V\varrho,\varrho)_{L_2(\Gamma)} = \int\limits_\Omega |\operatorname{grad} V\varrho|^2dx + \int\limits_{\mathbb{R}^3\setminus\Omega} |\operatorname{grad} V\varrho|^2dx\,,\; \varrho \in H^{-\frac{1}{2}}(\Gamma)$$

$$c_D\|\mu\|^2_{H^{\frac{1}{2}}(\Gamma)} \le (D\mu,\mu)_{L_2(\Gamma)} = \int\limits_\Omega |\operatorname{grad} W\mu|^2dx + \int\limits_{\mathbb{R}^3\setminus\Omega} |\operatorname{grad} W\mu|^2dx$$

for

$$\mu \in H_0^{\frac{1}{2}}(\Gamma) = \{\mu \in H^{\frac{1}{2}}(\Gamma) \mid \int\limits_\Gamma \mu do = 0\}$$

(see [9, 24, 51]).

Hence, $V^{-1} : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ is also well defined and continuous. The boundary energy norms now are defined by

$$\|\varrho\|_V := (V\varrho, \varrho)^{\frac{1}{2}}_{L_2(\Gamma)} \text{ and } \|\mu\|_{V^{-1}} := (V^{-1}\mu, \mu)^{\frac{1}{2}}_{L_2(\Gamma)}, \tag{4.12}$$

and they are equivalent to $\|\varrho\|_{H^{-\frac{1}{2}}(\Gamma)}$ and $\|\mu\|_{H^{\frac{1}{2}}(\Gamma)}$, respectively.

Then in [10] and [72] the following theorem was proved:

Theorem 4.3. *With* $q := \frac{1}{2} + \sqrt{\frac{1}{4} - c_V c_D} < 1$ *Neumann's operators have the following contraction properties:*

$$\|(\tfrac{1}{2}I_{(}\pm_{)}K)\mu\|_{V^{-1}} \le q\|\mu\|_{V^{-1}} \text{ for all } \mu \in H^{\frac{1}{2}}(\Gamma)\big(\mu \in H_0^{\frac{1}{2}}(\Gamma)\big),$$

$$\|(\tfrac{1}{2}I_{(}\pm_{)}K^*)\varrho\|_V \le q\|\varrho\|_V \text{ for all } \varrho \in H^{-\frac{1}{2}}(\Gamma)\big(\varrho \in H_0^{-\frac{1}{2}}(\Gamma)\big),$$

$$\|2K\mu\|_{V^{-1}} \le q'\|\mu\|_{V^{-1}} \text{ with } q' < 1 \text{ for all } \mu \in H_0^{\frac{1}{2}}(\Gamma).$$

As a consequence, Neumann's series with $A = L$ converge in the boundary energy spaces. Moreover, asymptotic stability and convergence of boundary Galerkin methods is valid for all the corresponding Dirichlet and Neumann problems [24]. For the realization of the scalar products in (4.12), however, one needs a Galerkin–Petrov method with test and trial functions in different spaces.

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